Order Statistic Filters as Postdetection Processors

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Abstract—**The performance of a radar detection system can be improved through postdetection processing. Two popular postdetection processors are the *n*-pulse and binary integrators. This paper examines the performance of the binary integrator from the viewpoint of order statistics and describes the binary integrator as an order statistic (OS) filter. A general analysis is developed for OS filters in detection systems and is used to show the OS filter is a consistent and biased estimator of the quantities of the received signal distributions. The features of the OS filter and *n*-pulse integrator that are critical to detection performance are compared to determine the proper application of each processor. It is shown that the OS filter can be used to emphasize selective regions of the input distributions where good statistical separability between the classes of input signals exist. The results from this analysis are useful for the development of optimization procedures over a variety of input signals and are applicable to any detection system where the sampled signals can be modeled statistically, such as in sonar and ultrasonic detection systems. A computer simulation is performed to illustrate the performance of binary and *n*-pulse integration for the detection of a white chi distributed target in white Weibull clutter and of a white Rayleigh distributed target in white Rayleigh clutter.**

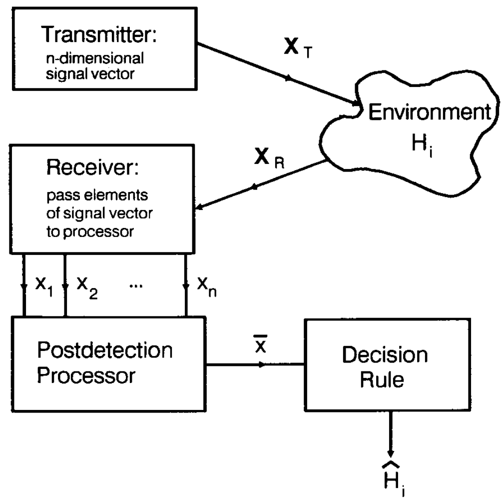

Fig. 1. The roll of the postdetection processor is illustrated in a general detection system. The transmitted signal vector X_T, made of n signal elements distributed over some domain (i.e., time, frequency, space, etc.), is received as X_R. The elements of the received vector, $x_1, x_2, \cdots, x_n$, are then processed to improve the detection performance of the decision rule (threshold comparison with $\bar{x}$ to decide target present H_1 or no target present H_0).

I. Introduction

The detection of a fluttering radar target in the presence of clutter and noise can be improved significantly via postdetection processing [1]–[4]. The function of the postdetection processor in the general detection system is illustrated in Fig. 1. Postdetection processors employed in the past have mainly consisted of n-pulse integration (pulse averaging) and binary integration [5]–[7]. The binary n-pulse integrator is the optimal processor for a fixed target in Gaussian noise, and for a Rayleigh power target in Rayleigh power clutter [7]. Binary integration is a nonlinear process that counts the number of times the return signal from a given sequence exceeds a threshold (referred to as the first threshold). When the sequence of return signals are received, the number of times the first threshold was exceeded is compared to another threshold (referred to as the second threshold) for the decision.

The binary integrator is considered to be a suboptimal processor [5], [6]. The main advantage of the binary integrator is that previous radar pulses do not need to be stored for averaging after the sequence. Though signal storage is not a major concern for modern radar, binary integration can offer other distinct advantages over n-pulse integration when non-Rayleigh target and clutter distributions are encountered. This paper examines the performance of binary integration from the viewpoint of order statistics [8], in which the binary integrator is presented as a special case of an order statistic (OS) filter.

The OS filter is a digital filter that operates on n input signals to generate an output, $x_{r:n}$, such that $x_{r:n}$ is equal to one of the input signal values that is less than or equal to $n - r$ input values and greater than or equal to $r - 1$ input values. The OS filter can be expressed by

$$x_{r:n} = OS_{r:n}\{x_i, x_{i+1}, x_{i+2} \cdots x_{i+n-1}\} \quad \text{for } 1 \le r \le n \tag{1}$$

where x_i is the unordered observed input value from the sequence, n is window size of the filter, and r is the rank of the input from the ordered sequence that becomes the output $x_{r:n}$. This filter is the median detector when $r = (n + 1)/2$ (for n odd), the maximum detector when $r = n$, and the minimum detector when $r = 1$.

Order statistic filters have been applied in noise power estimation of sonar signals [9] for rejection of bad data

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(outliers). They also have been applied for processing image [10], [11] and speech [12] signals because of their noise suppression and feature preservation properties. The application to image and speech processing has given rise to special signal representations for OS filter analyses [13], [14]. One notable analysis of the OS filter is presented in [15], where it is considered a special case of a morphological filter.

The analysis developed in this paper considers the application of OS filters as a postdetection processor in various target and clutter environments. The results of this analysis are directly applied to the detection problem, but they are general enough for application to any problem where at least partial information concerning the distribution function of the signal and noise is known. It is shown that the OS filter can be used for suppressing certain types of statistical behavior of the input signal, while passing other types of behavior characterized by selective regions of the input distribution function.

The OS filter is equivalent to the binary integrator when a threshold is applied to the output of the OS filter. The only difference between them is the order in which the two thresholds are applied. The equivalent between these two processors is understood by the property that the OS filter operation commutes with a thresholding operation of the signal [14], [15]. To show this, let the first threshold of the binary integrator be η_b and the threshold at the output of the OS filter be η_o. Also, let the second threshold of the binary integrator be r_b and the rank of the OS filter be r_o. Now if $\eta_b = \eta_o$ and the output of the OS filter $x_{r:n}$ exceeds this threshold, then at least $n - r_o + 1$ of the n values must also exceed this threshold. Likewise for the binary integrator, if r_b values of the sequence exceed η_b, then input value with rank $n - r_b + 1$ must also exceed the threshold. Therefore, if $\eta_o = \eta_b$ and $r_o = n - r_b + 1$, the OS filter and the binary integrator yield the same decision for each set of input signals.

The first statistical analysis for n-pulse integration was performed by Marcum in 1947 [1] for a fixed target in white Gaussian noise. In 1954, more realistic target models were applied by Swerling [2]. These models consisted of distributions from the chi-square family, which accounted for the fluctuations in the target returns. These were both considered pioneering work in statistical analysis of radar performance that opened the door for more performance analysis involving clutter distributions such as Weibull [16], lognormal [17], and compound models [18]. A statistical analysis for binary integration was performed by Harrington [3], which considered a fixed target in white Gaussian noise and determined the optimal thresholds η_b and r_b. Linder and Swerling also investigated binary integration [19] to find the optimal threshold r_b for a Rayleigh fluctuating target, which they found to be $r_b = 6$ when n equaled 10. Later on Trunk and George [17] investigated the detection of a fixed target in lognormal clutter and found the binary integration to be superior to n-pulse integration. Recently, Ward [18] examined the performance of binary integration for fixed targets in partially correlated clutter. The results showed that the binary integrator performed slightly better than the n-pulse integrator.

This paper develops a statistical analysis of the OS filter to obtain general input–output relationships so that predictions concerning the performance of the binary integrator in various target-clutter situations can be made. The relationship between the optimal second threshold r_b and properties of the target and clutter distributions are discussed. Section II examines the expected value of the output of the OS filter for indpendent and identically distributed input signals. A new derivation is presented that shows the OS filter as an estimator of the quantiles of the input signal distribution. Section III compares the features of the OS filter and n-pulse integrator and explains how these properties can be used to improve detection performance. Section IV discusses a computer simulation that compares the performance of binary and n-pulse integration for the detection of a white chi distributed target in white Weibull clutter and a white Rayleigh distributed target in white Rayleigh clutter. Finally, Section V comments on the noise suppressing properties of this filter and its application when only partial information concerning the signal and noise distribution are available.

II. The OS Filter as a Quantile Estimator

A quantile is a value from a set of values that divide the distribution into equal probability regions. If the areas under the probability density function between all pairs of consecutive points (including the end points of the distribution) are equal, these points are quantiles. In this section, the sort function is introduced for the analysis of OS filters. In [8] it was shown that the OS is an asymptotic estimator of the quantile, but in this section the properties of the sort function are applied to show this result. It also is shown that in general this estimator is biased.

Given that the input signal of the OS filter x_i are independent and identically distributed with distribution function $F_X(x)$, the probability density function for the output is given by a well-known result in order statistics [8]

$$f_{X_{r:n}}(x) = r\binom{n}{r} F_X^{r-1}(x)\left(1 - F_X(x)\right)^{n-r} f_X(x) \quad \text{for } 1 \le r \le n \tag{2}$$

where r and n are defined as in (1), and $f_X(x)$ is the input probability density function. Throughout this paper x represents the real value of the observations of the input sequence and $x_{r:n}$ is the real value of ordered sequence, while X is the random variable for the input of the OS filter and $X_{r:n}$ is the random variable for the ordered sequence of the OS filter.

The density function for $X_{r:n}$ in (2) is the product of the probability density function of a single input $f_X(x)$ and another function given by

$$W(x) = r\binom{n}{r} F_X^{r-1}(x)\left(1 - F_X(x)\right)^{n-r} \quad \text{for } 1 \le r \le n. \tag{3}$$

While the exact behavior of the output of the OS filter depends on $f_X(x)$, the general behavior of this filter, particularly the mean and variance of the output, can be determined by the properties of $W(x)$.

Before finding the expected value of the output of the OS filter, the notation can be simplified by letting $u = F_X(x)$. To obtain corresponding values in the x domain (values from the original input distribution) from u, the inverse of the distribution function can be used, denoted by

$$x = F_X^{-1}(u) \qquad \text{for } 0 \le u \le 1. \tag{4}$$

The set of quantiles in the x domain correspond to points in the u domain that divide the u axis (from 0 to 1) into equal intervals.

If u is substituted into (3), the result is

$$w_{r:n}(u) = r\binom{n}{r} u^{r-1}(1-u)^{n-r} \qquad \text{for } 0 \le u \le 1 \tag{5}$$

where the subscript r: n denotes the parameters of the associated OS filter. This function by itself is the beta probability density function [20], [21]. For its application in this paper it will be referred to as the *sort function*. The modal point of $w_{r:n}(u)$ occurs when u is equal to a value, which will be denoted by t given as [21]

$$t = \left(\frac{r-1}{n-1}\right) \qquad \text{for } 1 \le r \le n. \tag{6}$$

The symbol t will be used throughout this paper to denote the value of u where $w(u)_{r:n}$ is a maximum. An alternative notation for the sort function will use the t parameter in place of the r parameter written as $w(u)_{t:n}$. This will be useful later for analysis when the modal point of the sort function is kept constant and n is increasing.

For a given n, the set of t values corresponding to all the possible values of r constitute a set of quantiles for the input distribution. The extreme end points of the distribution corresponding to $u = 0$ and $u = 1$, will be referred to as the 0th quantile and the $(n-1)$th quantile, respectively. In Fig. 2 the plots of the sort function for five different values of r are presented with $n = 5$. In each plot note that the set of modal points divide the u domain into four equal intervals. Therefore, these modal points correspond to quantiles through the inverse distribution function.

Consistent Estimator: The sort function in (5) is used now to show that the OS filter is a consistent estimator of the quantile values for increasing n and constant t. The expression for the mean of the output of the OS filter is given by

$$E[X_{r:n}] = r\binom{n}{r}\int_{-\infty}^{\infty} x f_X(x)\, F_X^{r-1}(x) \cdot \left(1 - F_X(x)\right)^{n-r} dx \qquad \text{for } 1 \le r \le n. \tag{7}$$

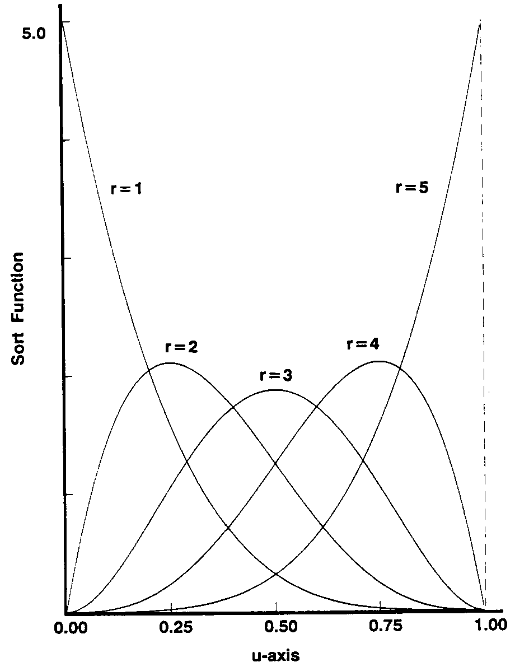

Fig. 2. Five different sort functions for window size $n = 5$ demonstrate the relationship between the quantiles and the modal points determined by the rank parameter r.

If the u substitution is performed on (7), the result is [8]

$$E[X_{r:n}] = \int_0^1 F_X^{-1}(u)\, w_{r:n}(u)\, du. \tag{8}$$

This equation reveals that the expected value of the output is the integral of the product between the sort function and the inverse distribution function. Fig. 3 shows these two functions superimposed on one another. Note how the sort function acts as a weighting function to emphasize a particular region of the distribution function over the integration. By changing the r parameter of the sort function, the modal point can be set to emphasize different regions of the distribution. For increasing n, with t held constant, the sort function is a delta sequence shifted right on the u axis by an amount equal to t. A proof for this is shown in Appendix A. Now for a constant t as n approaches infinity, (8) becomes

$$\lim_{n\to\infty} E[X_{t:n}] = \int_0^1 F_X^{-1}(u)\, \delta(u-t)\, du \tag{9}$$

where $\delta(\cdot)$ is the Dirac delta function [22]. At this point both n and r approached infinity, but t remains a finite ratio of n and r. Equation (9) implies

$$E[X_{t:\infty}] = F_X^{-1}(t). \tag{10}$$

Since t corresponds to the endpoints of the equal subdivisions along the u axis, the inverse distribution function of t is the $(r-1)$th quantile for a particular n satisfying (6).

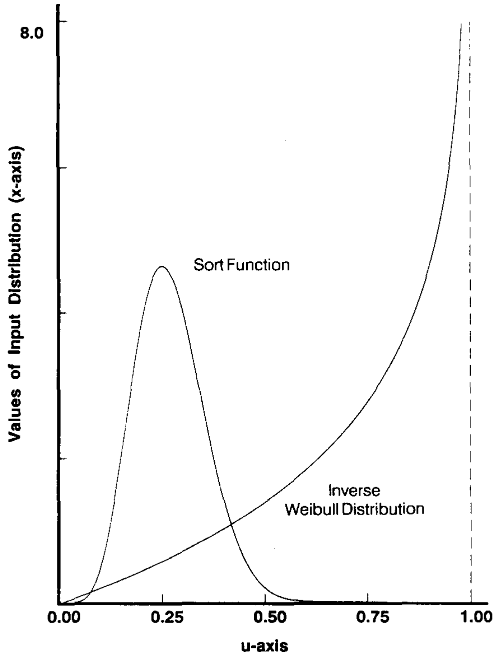

Fig. 3. The sort function with $n = 25$ and $r = 7$ is superimposed on an inverse Weibull distribution function. The integral of the product between these two functions is the expected value of the corresponding OS filter output. This illustrates how the expected values of the output is affected by changing the rank to emphasize specific regions of the distribution.

It now has been shown, via the properties of the sort function, that the OS filter is a consistent estimator of the quantile value corresponding to a constant t. The sort function with a finite n will have some dispersion about the modal point that allows the values in the neighborhood of the quantile to influence the output. This will result in a possible bias in the estimate of the quantile for a finite n.

Biased Estimator: In the previous discussion, it had been shown that for increasing n the expected value of the output approaches the quantile values of the input distribution, but to obtain the actual expected value of the output for finite n, the bias level of the output must be examined. This bias cannot be determined in general, since the OS filter is nonlinear and the input distribution affects the outcome. Therefore, the sort function is examined to indicate the tendencies of the filter when used over a general class of input distributions. Explicit results are derived for the case when the input distribution is uniform, which can also be found in [8], and these results are applied to develop a linear approximation for the general case.

If the input is uniformly distributed between zero and a, the inverse distribution function becomes

$$F_X^{-1}(u) = au \qquad \text{for } 0 \le u \le 1. \tag{11}$$

Since u ranges between 0 and 1, a is also equal to the constant slope of the inverse distribution function. If (11) is substituted into (8), the expected value of the output becomes [8]

$$E[X_{r:\infty}] = \frac{ar}{n+1} \qquad \text{for } 1 \le r \le n. \tag{12}$$

The modal point of the sort function for the given n and the r in the u domain is given by (6). The corresponding value in the x domain is found by substituting t, in terms of n and r, into (11). The difference between the expected value and the modal point (same as quantile value) is

$$E[X_{r:n}] - at = \frac{a(n+1-2r)}{n^2+1} \qquad \text{for } 1 \le r \le n. \tag{13}$$

The sign of the bias changes when r passes through the value $(n + 1)/2$. A positive bias implies that the expected value is to the right of (or greater than) the modal point. A negative bias implies the expected value is to the left of (or less than) the modal point. The rank parameter r was used to adjust the skewness of the sort function, from right to left, as r is increased. Skewness indicates that more emphasis is on one side of the quantile value than the other. It also is observed from (13) that increasing n results in a decreased bias level. This should be expected from the fact that the estimator is consistent. The power of the input is related to the scaling parameter a. This indicates that as the input power is increased the bias level also is scaled in magnitude.

When the input is not uniformly distributed, (13) does not apply, but if a neighborhood in the region of emphasis of the sort filter is observed to be nearly linear (see Fig. 3), then a linear approximation can be made. The value of the slope in the region of emphasis can be used for the value a in (13). For a large enough n (typically greater than 10, for most exponential type distributions) the slope of the inverse distribution function evaluated at the mean or modal point of the sort function, should give a good representative slope for that region.

Another useful quantity for determining the behavior of the output of the OS filter is the variance. The second moment about an arbitrary point corresponding to u_0 is written as

$$M_{2,u_0} = \int_0^1 \left(F_X^{-1}(u) - F_X^{-1}(u_0)\right)^2 w_{t:n}(u)\, du. \tag{14}$$

For the uniform distribution between zero and a, (14) becomes:

$$M_{2,u_0} = \frac{a^2(u_0(1-u_0)(n-7)+2)}{(n+2)(n+1)}. \tag{15}$$

If u_0 is equal to the mean value given in (12), then the above expression is the variance for the uniform distribution. This quantity can also be applied as a linear approximation of the variance for a general distribution function by letting a equal the approximate slope within the region of emphasis of the sort function. When a is unity, then the above expression is the variance of sort function itself. Note that the variance decreases at a rate proportional to $1/n$, except when u_0 is equal to 0 or 1, then the second moment decreases at a rate proportional to $1/n^2$. These observations are consistent with the fact that the sort function is a delta sequence.

III. Comparison of the OS Filter and N-Pulse Integrator as Postdetection Processors

In this section, the performance of the OS filter and the n-pulse integrator are compared. Both processors are considered feature extractors of the input signal distribution. Therefore, the performance of the processors are discussed in terms of feature selection. One measure used to determine the quality of a feature, based on the first two moments, is Fisher's criterion [23]. This measure is a ratio given by

$$F = \frac{(\mu_1 - \mu_0)^2}{\sigma_1^2 + \sigma_0^2} \tag{16}$$

where μ_1 and μ_0 are the mean values of the output feature from the two different classes of signals that are being detected, and σ_1^2 and σ_0^2 are the variances of the output feature. This quantity is a measure of the distance between the means of the two feature distributions normalized by their variances. Therefore, a high F value indicates a good feature and implies that this feature can be used to discriminate between the two classes of input signals with low probability of error.

The n-pulse integrator estimates the mean of the input distribution. Therefore, the expected value of the input will be that of the output feature. From Fisher's criterion it can be understood that the n-pulse integrator should be applied when the means of the two input signal classes are distinct. The greater the distance (in terms of the variances) between the mean values, the better the n-pulse integrator is expected to work. The variance of the output of the n-pulse integrator is equal to that of the input reduced by a factor of $1/n$. This reduction in variance also causes the value of the Fisher's criterion to increase.

For many applications the n-pulse integrator is a good choice for a postdetection processor, since it is guaranteed that the variance of the output feature will be reduced by a factor of the reciprocal number of independent samples processed. The n-pulse integrator reduces the probability of error in detection due to the fluctuations of the individual samples by estimating the mean of the input distribution. However, for applications where the means of the two classes of signals are very close in value, the performance of the n-pulse integrator is limited. In these situations the OS filter can offer distinct advantages over the n-pulse integrator.

For a given sequence of n samples the OS filter has the freedom to generate n features, which are the n different quantile values. The advantage of this freedom is that it allows the processor to focus on particular regions of the distribution functions where significant differences exist.

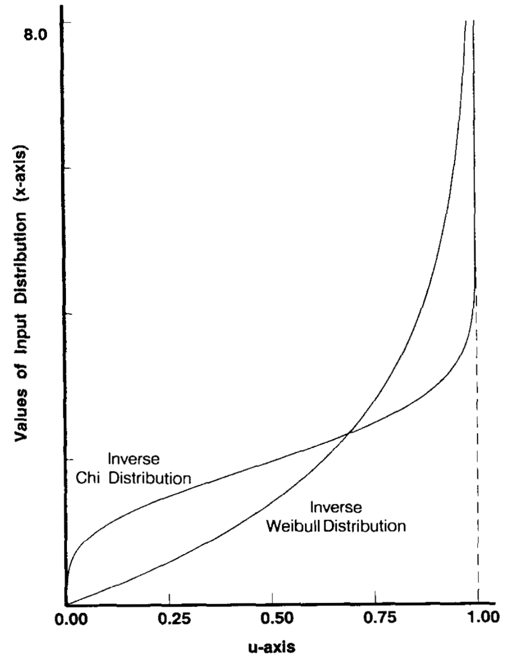

Fig. 4. The inverse Weibull distribution function with shape parameter equal to 1.0 and inverse chi distribution function with 4 degrees of freedom, both with identical mean values, are plotted together.

This can occur when one distribution of a class of input signals is more skewed than the other. This is illustrated in Fig. 4. The inverse chi and Weibull distributions are plotted together. Both distributions have equal mean values, yet regions exist where the distributions are more distinct from each other. These are the regions that should be emphasized by the OS filter by setting the rank parameter to estimate the proper quantile. In this case the lower quantile and the upper quantile regions indicate a good separation.

A separation between regions of the inverse distribution functions, as described by (16), involves both the distance and slope difference between the functions at the particular quantile value. Recall that for finite n a bias exists that is dependent on the slope in that region (see (13)). Another factor in searching for a good quantile feature is a small output variance. This quantity is also dependent on the slope of the inverse distribution function in the region of emphasis as noted in (15). These factors are taken into consideration to determine the best quantile feature through Fisher's criterion.

It is predicted that the OS filter will outperform the n-pulse integrator when a shape difference exists between the probability distributions of the clutter and the target-plus-clutter signals at low signal-to-clutter ratios (SCR). The shape difference can be defined in terms of the inverse distribution functions as

$$\Delta s = \int_0^1 \left(F_1^{-1}(u) - aF_0^{-1}(u)\right)^2 du \tag{17}$$

where a, chosen to minimize Δs, is equal to

$$a = \frac{\int_0^1 F_0^{-1}(u)\, F_1^{-1}(u)\, du}{\int_0^1 \left(F_0^{-1}(u)\right)^2 du}. \tag{18}$$

If Δs is zero, then no shape difference exists between the two distributions.

In the case of no shape difference, the n-pulse integrator is predicted to work better, because the major statistical parameter for discrimination is a scale factor, which affects all quantile regions proportionately and is well represented by the mean value. When a shape difference exists and the signal-to-noise ratio (SNR) or SCR is high enough such that $F_1^{-1}(u) \geq F_0^{-1}(u)$ for all u, the n-pulse integrator also is predicted to perform slightly better than the OS filter, since all the quantiles regions are affected and the mean values utilizes the information in all these regions.

As the SCR decreases, the decision on whether to use the OS filter or the n-pulse integrator is more critical. When a shape difference exists and the SCR becomes small enough, such that the separation between the means of the two distributions is smaller than an individual quantile separation (separation as defined in (16)), the OS filter can be applied to obtain superior performance over the n-pulse integrator by emphasizing that quantile region over the others. This is illustrated by simulations in the next section.

IV. Performance of OS Filter and N-Pulse Integration for a Chi Distributed Target in Weibull Clutter

In this section the performance results from a simulation of the OS filter and the n-pulse integrator are presented. Two ensembles of signals were generated for each simulation. One set of signals generated from clutter only were obtained from a highly skewed Weibull distribution whose probability density function is given by

$$f(x) = \frac{cx^{c-1}}{b^c} \exp\left(-\left(\frac{x}{b}\right)^c\right) \qquad \text{for } x \geq 0 \tag{19}$$

where b is related to the power in the return wave (referred to as the scale parameter) and c is the shape parameter that controls the skewness or the tail of the distribution. In the first simulation the shape parameter was chosen to be equal to 1.0. The signals from the target only returns were obtained from the chi distribution, whose probability density function is given by

$$f(z) = \frac{2z^{N-1}}{(\sigma\sqrt{2})^N\, \Gamma\left(\frac{N}{2}\right)} \exp\left(\frac{-z^2}{2\sigma^2}\right) \qquad \text{for } z \geq 0 \tag{20}$$

where N is the number of degrees of freedom. For the first simulation N was chosen to be 4.

To simulate the target-plus-clutter signal, a simple addition of the clutter and target values will not be accurate, since the sum does not take into account the constructive and destruction interference due to the random phase difference between the two return waves. The interference can be accounted for by considering the new amplitude resulting from the addition of two sinusoids of different amplitude and phase. Let X be the random variable representing the amplitude of the clutter only return, Z the random variable representing the amplitude of the target only return, and θ a uniformly distributed random variable between 0 and π representing the phase difference between the clutter and target return, then the resultant amplitude of both waves is given by:

$$Y = \sqrt{X^2 + 2XZ\cos(\theta) + Z^2}. \tag{21}$$

The cross-product term in (21) represents the constructive and destructive interference of the returning signals due to phase differences between the target and the clutter scattering centers. When a large SCR exists between the clutter and target returns, this calculation is not very critical; the shape of the target-plus-clutter distribution is close to that of the target alone. But as the SCR is reduced between the target and clutter, the shape of the target-plus-clutter distribution approaches that of the clutter. This shape transition is accounted for in (21).

The SCR referred to in this simulation is the ratio of the second moments of the target alone over the clutter alone distributions. In the simulation the minimum required SCR was determined to obtain a given probability of false alarm and detection for a decision based on one sample. Then the SCR was decremented by 1 dB and the window size of the filter n was incremented until the desired false alarm and probability of detection were obtained.

For each increment of n, a new output distribution function for the clutter only returns was obtained by generating samples to be processed by OS filtering or n-pulse integration. From this distribution function the proper threshold for the false alarm probability was found. The probability of detection was checked by generating samples of the random quantity in (21) and testing them with the previously obtained false alarm threshold. The plots obtained from the simulation are referred to as the required SCR plots where the abscissa is the window size of the filter n, and the ordinate is the minimum SCR for the target and clutter returns required to obtain the given probability of false alarm and detection. The simulation terminated when the probability of detection no longer improved for increasing n. This occurred for low SCR when the quantiles (or means in the case of n-pulse integration) were so close that the variance reduction due to the processing was not sufficient to improve the probability of detection.

Fig. 5 is a plot of the required SCR with a false alarm probability of 10^{-3} and a probability of detection equal to 0.5 for the OS filter and the n-pulse integrator. The OS filter was tested for five different t values evenly spaced over the u axis, starting with $t = 0.0$ and ending with t

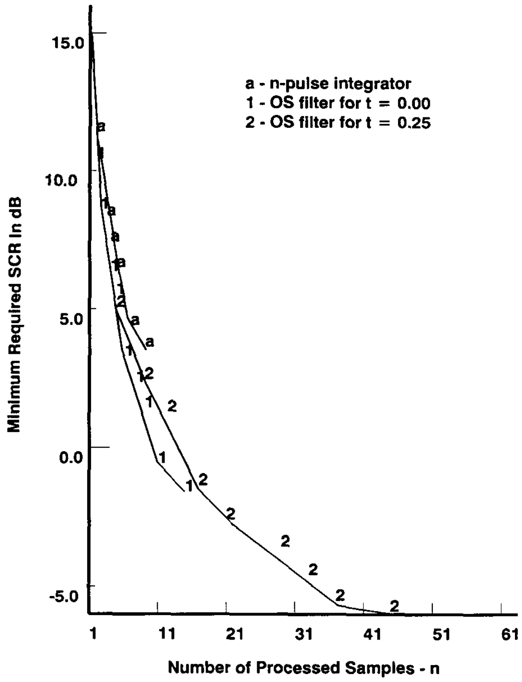

Fig. 5. The minimum SCR for a probability of false alarm equal to 10^{-3} and probability of detection equal to 0.5 is plotted versus the number of independent samples processed for the n-pulse integrator and two OS filters focused on different quantile regions. The returns for the clutter return signals were simulated for a highly skewed Weibull distribution with shape parameter equal to 1.0 and the target returns were simulated for a chi distribution with 4 degrees of freedom. This illustrates the advantage of the OS filter over the n-pulse integrator.

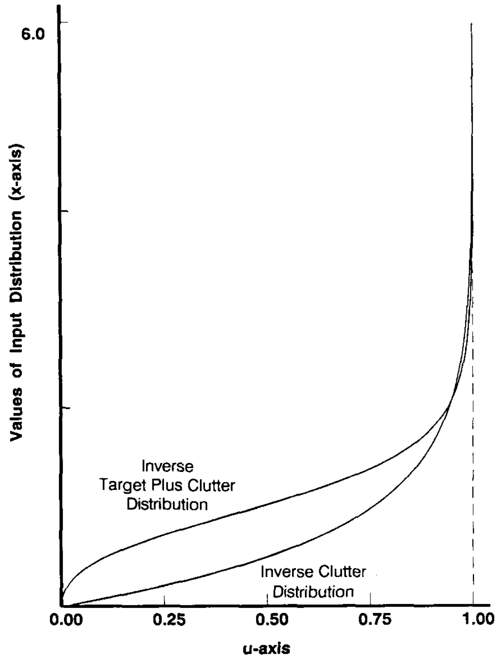

Fig. 6. The inverse distribution function for the clutter and target-plus-clutter returns for the simulation of Fig. 5 are plotted together with a 0-dB ratio between the second moments of the clutter only and the target only return signals. Note that the presence of the target causes a greater separation in the lower quantile regions.

= 1.0. The performance of the OS filter corresponding with the t values that yield the two best performances are plotted. In this case it turned out that t corresponding to the lower quantiles performed the best; $t = 0.0$ and $t = 0.25$. It is significant to note that for the high SCR the performance of the OS filter and n-pulse integrator is approximately the same. As the SCR decreases below 5.0 dB, the OS filter demonstrates a significant superiority in performance over the n-pulse integrator. At the expense of 5 additional samples the minimum detector was able to meet the detection requirements at about 6.0 dB less than the lowest SCR required for n-pulse integration. Beyond this point the efficiency of the OS filter decreased sharply, though at the expense of 25 more samples the OS filter corresponding to $t = 0.25$ was able to meet the detection requirements at 4.0 dB below that possible by the minimum detection (a total of 10 dB below that of the n-pulse integrator).

Results of Fig. 5 are consistent with the predictions made in Sections II and III. The behavior of these post-detection processors at low SCR can be understood by examining the inverse distribution functions for the target-plus-clutter and clutter-only signals. Fig. 6 is a plot of these inverse distributions with a 0.0 dB SCR ratio. Note the poor separability between the two distributions near the upper quantiles. This is due to the skewness of the clutter distribution. This implies that the data from values in this region can be misleading, since the clutter distribution in this region has more power than the target plus clutter. These values cause a degradation in the performance of the n-pulse integrator, which utilizes all data values in the averaging operation. The OS filter can censor the data returns from this region and emphasize the data from the regions with better separability, which exist toward the lower quantiles.

Fig. 7 is the required SCR plot for a Rayleigh distributed target in Rayleigh distributed clutter. In this case the performance of the n-pulse integrator is superior to that of the OS filter. The t values corresponding to the two best performances of the OS filter correspond to the middle to upper quantile ranges; $t = 0.5$ and $t = 0.75$. Above a 5.0-dB SCR the performance of the OS filter and the n-pulse integrator is similar, with the n-pulse integrator performing slightly better. For SCR values below 5.0 dB, the efficiency of the n-pulse integrator is clearly superior to that of the OS filter. At 21 samples, the n-pulse integrator is able to detect a target at 2.0 dB below that of the OS filter. At the expense of 20 to 30 more samples, the OS filter was able to achieve the given detection criterion at the same minimum SCR required for the n-pulse integrator.

The results in Fig. 7 are also consistent with the predictions made in Sections II and III. In Fig. 8 the two inverse distribution functions are plotted for the clutter and target-plus-clutter return signals. The SCR between

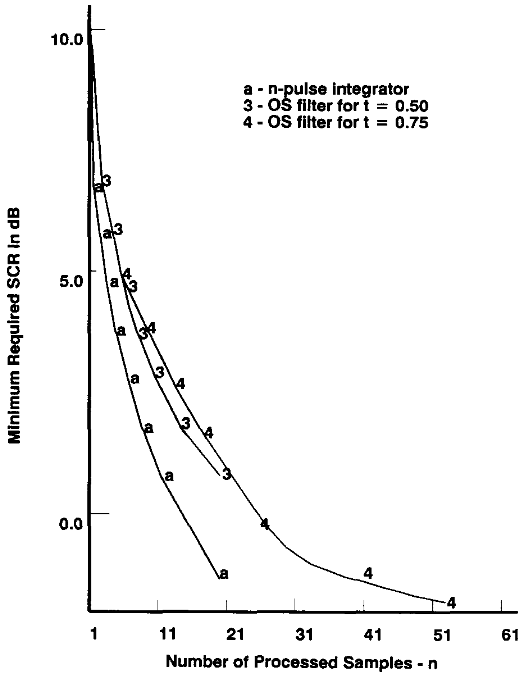

Fig. 7. The minimum SCR for a probability of false alarm equal to 10^{-3} and probability of detection equal to 0.5 is plotted versus the number of independent samples processed for the n-pulse integrator and two OS filters focused on different quantile regions. The returns for the clutter and target return signals were both simulated for a Rayleigh distribution. This illustrates the effectiveness of the n-pulse integrator when no shape difference exists between the target and clutter fluctuations.

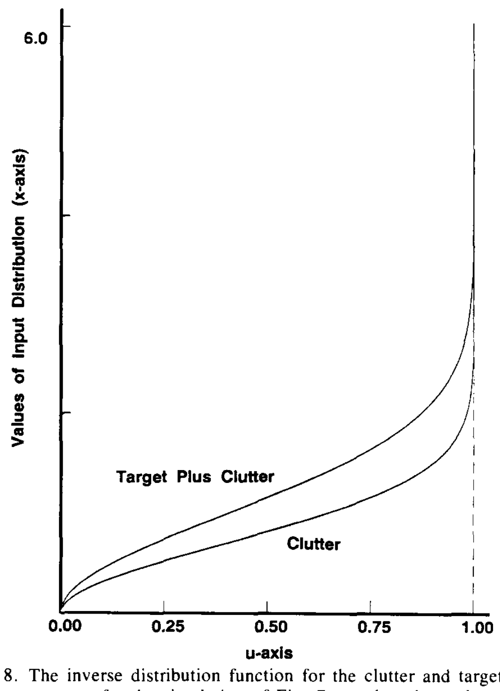

Fig. 8. The inverse distribution function for the clutter and target-plus-clutter returns for the simulation of Fig. 7 are plotted together with a 0-dB ratio between the second moments of the clutter only and the target only return signals. Note that the presence of the target results in good separation in all quantile regions.

the target and clutter return signals is 0.0 dB. In this case good separability exists over the whole range of quantiles and the target-plus clutter inverse distribution is always greater than the clutter-only inverse distribution. This implies that data samples from all regions of the distribution contribute significant information to the decision. The OS filter does not utilize the information contained in all samples as does the n-pulse integrator. Therefore, the n-pulse integrator's performance is superior to the OS filter's. The best regions of operation for the OS filter were the middle to upper quantile regions, where the greatest separation between the two distributions is observed.

V. Application of OS Filter with Limited Knowledge of Target and Clutter Distribution

The simulations in Section IV have demonstrated the advantages of OS filtering over averaging when shape differences exist between the distribution functions. This observation also is supported by the sort function analysis, given the *a priori* knowledge of the clutter and target signal distribution. In practice, such complete *a priori* knowledge is typically not available in a practical problem. Therefore, in this section, examples of a shape difference occurring between distribution functions is considered based on more general knowledge of the clutter and target signals.

Consider the case where the clutter signal is an echo from a group of randomly distributed scatterers throughout the resolution cell, and the target signal is an echo from a scatterer with greater reflectivity than that of the individual scatterers of the clutter. In this case the clutter and target-plus-clutter signals will fluctuate over time as scatters move with respect to the signal transmitter/receiver, or, if the scatters remain stationary, random fluctuations can occur over frequency variations of the signal. The critical question to ask in applying the OS processor is, "If a target scatterer appears in the resolution cell, in what quantile region is the strongest change (i.e., shape of distribution function) most likely to occur?"

The quantile region most sensitive to the occurrence of a target scatter can be determined analytically in some cases. For example, if the clutter scatterers are such that they give rise to Rayleigh distributed fluctuations [24], [25], and a target scatterer, whose fluctuations are steady, enters the resolution cell, then it has been shown that the target-plus-clutter distribution will have a Rayleigh–Rice distribution [24], [25]. In this case the sort function analysis can be applied directly, and the optimal rank can be determined from Fisher's criterion (16). In cases where derivation of the target-plus-clutter distribution is not possible, simulations can be performed to find the inverse distribution, as in the previous section (see Figs. 6 and 8).

If the nature of the fluctuations of the clutter and the target is known, the best performing ranks of the OS filter can be reasoned from the sort function analysis. For example, if the scatterers generate specular reflections (spiky signals), the pdf of the signal will have a larger tail (relative to that of the Rayleigh distribution) [26], [17]. An increase in the tail of the pdf results in an increase of the values in the upper quantile regions of the inverse distribution function. Therefore, if the target is specularly reflective (more so than the clutter), the upper quantile regions are of more interest.

If the clutter was more specularly reflective than the target, the lower quantile regions are of more interest. This latter case occurred in many ultrasonic nondestructive evaluation experiments (NDE) with flaws embedded in large grained steel samples. The clutter was analogous to the grain boundaries in the steel, and a split-spectrum (frequency variation of pulse-echo signal) technique [28] was used to obtain fluctuations in the return echo. The clutter echos fluctuate according to the Rayleigh distribution, while the target (in many experiments was a flat bottom hole) fluctuates significantly less than the clutter echos. In these cases the minimum detector (OS filter with rank equal to 1) performed better than the averaging techniques. This result is expected from the sort function analysis. In some recent experiments [29], using irregular surfaces for target or flaw, the higher rank values performed better than the minimum and lower rank order statistics. This was the result of the target echos fluctuating in a more specular manner than the clutter.

When very little is known about the nature of the clutter and target, one can extend the ideas generated by the sort function analysis to derive adaptive or robust schemes. For example, the first two moments of all the order statistics can be measured, stored, and updated while the radar scans the environment. Thresholds can be set adaptively for all order statistics, and then thresholds exceeded in various quantile regions can indicate the appearance of a target and the nature of its reflections (i.e., specular, steady, or equivalent to that of the clutter). For robust processing, the order statics that exhibit the greatest stability over time can be chosen for reliable detection. This is often the median value [17], [30], since it is typically more difficult to characterize the tail of the distribution from real data.

The decision of whether to use the OS filter or n-pulse integrator can also be made from the *a priori* knowledge of the fluctuations of the target and clutter. If the target is a cluster of scattering centers similar to that of the clutter (except that the target scatters have a higher reflectivity), then the target-plus-clutter and the clutter signal will tend to have the same distribution. Therefore, each quantile region is important and n-pulse integrator should be used.

VI. Conclusions

This paper has shown the performance of the OS filter to be equivalent to that of the binary integrator. Therefore, the results derived for the OS filter are directly applicable to the design and application of the binary integrator.

It was established through the sort function analysis that the OS filter is a consistent, biased estimator of the quantiles of the input distributions. This was used as a basis for a comparison between the performance of the OS filter and n-pulse integrator, which estimates the mean value of input. From this comparison it was concluded that the n-pulse integrator performs better than the OS filter when the shapes of the distribution functions of both classes of signals are similar, or the statistical fluctuations target are of the same distribution as the clutter (only differing by a scaling parameter). In applications where a shape difference between the two distribution functions existed, the OS filter outperforms the n-pulse integrator by its ability to focus on quantile regions of good separation between target-plus-clutter and clutter-only distribution.

The form of the output probability density function for the OS filter revealed that the output statistics were the result of the integral of a product between the inverse distribution function and a function referred to as the sort function. The relationship between the filter parameters and the properties of the sort function was established. It was then shown that the sort function could be used to emphasize quantile regions of the distribution function where the greatest separation (as defined by Fisher's criterion) between the distribution functions occurred due to the presence of a target. The results from a computer simulation demonstrated consistency between the predictions made from the sort function analysis and the simulated results.

Appendix A

This Appendix proves that the sort function of (5) is a delta sequence for increasing n. If t is held constant as n approaches infinity, it is shown that the sort function converges on the delta function denoted by $\delta(u - t)$. To establish this, it is sufficient to show that [22]

$$\int_0^1 w_{t:n}(u) = 1 \qquad \text{for all } n \geq 1 \tag{A.1}$$

and that

$$\lim_{n \to \infty} w_{t:n}(u) = \begin{cases} \infty; & u = t \\ 0; & u \neq t. \end{cases} \tag{A.2}$$

Since the limit as n approaches infinity is examined while t is constant, the t parameter is referred to rather than the varying r parameter. The condition stated in (A.1) is satisfied since $w_{t:n}(u)$ is the beta probability density function [20], [21]. The conditions in (A.2) can be proved by considering two cases, where case one is for $t = 0$ or 1 and case two is for $0 < t < 1$.

Case 1: Consider $t = 0$. This implies that $r = 1$ and hence, the sort function in the limit of (A.2) reduces to

$$\lim_{n \to \infty} w_{0:n}(u) = \lim_{n \to \infty} n(1 - u)^{n-1}. \tag{A.3}$$

This limit approaches zero when the quantity raised to the exponent is less than unity. This is satisfied for $u \neq 0$. When $u = 0$, the exponential term has no effect (always equal to unity) and the limit approaches infinity. When t

$= 1$, which implies $r = n$, an analogous argument to the one just given for $t = 0$ can be used to show the desired result.

Case 2: Now consider $0 < t < 1$. Since n is approaching infinity, the Sterling approximation for factorials can be used. If this approximation is substituted into the factorial terms of the sort function, the limit in (A.2) can be written as

$$\lim_{n\to\infty} \frac{K\sqrt{n}}{(t)(1-t)}\left[\frac{n}{n-1}\right]^n\left[\frac{u^t(1-u)^{1-t}}{t^t(1-t)^{1-t}}\right]^{n-1} \tag{A.4}$$

where K is a finite constant. When $u = t$, the far right factor in the above expression becomes unity and the limit reduces to

$$\lim_{n\to\infty} \frac{K}{(t)(1-t)} \frac{[n/n-1]^n}{1/\sqrt{n}} = \infty \qquad \text{for } u = t. \tag{A.5}$$

The above limit is true because the numerator approaches the finite constant e and the denominator approaches zero.

When $u \neq t$, it can be shown that the far right factor of (A.4) is always less than unity:

$$\alpha = \frac{u^t(1-u)^{1-t}}{t^t(1-t)^{1-t}} < 1 \qquad \text{for } u \neq t. \tag{A.6}$$

In the above equation, the maximum value for α is unity and occurs at $u = t$. But since the situation considered here is for $u \neq t$, the inequality in (A.6) holds. Now with α defined as in (A.6), (A.4) by rearrangement of factors can be written as

$$\lim_{n\to\infty} \frac{K}{(t)(1-t)} \frac{\sqrt{n}\,[\alpha]^{n-1}}{[n/n-1]^{-n}} = 0 \qquad \text{for } 0 \le \alpha < 1. \tag{A.7}$$

Note that the denominator approaches the finite value e^{-1} and the numerator approaches zero since $0 \le \alpha < 1$. This verifies the limit given in (A.7).

It can now be stated that $w_{t:n}(u)$ is a delta sequence for n approaching infinity

$$\lim_{n\to\infty} r\binom{n}{r} u^{r-1}(1-u)^{n-r} = \delta(u-t) \qquad \text{for } t \text{ constant}. \tag{A.8}$$

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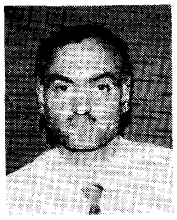

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