# Correspondence

## Statistical Evaluation of Sequential Morphological Operations

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*Abstract*—**In order to properly apply sequential morphological operations to random signals in applications concerned with noise suppression, we have examined their statistical properties using different structuring elements. The performance of flat and triangular structuring elements has been evaluated for signals with uniform, Gaussian, and Rayleigh density functions. In particular, the statistical properties of sequential morphological operations (i.e., dilation, closing, clos-erosion, and clos-opening) are examined as a function of the parameters of the structuring element through Monte Carlo simulation, which overcomes the statistical dependency problem arising in the processed signal at different stages of morphological operations. The simulated results and their statistics (mean, variance, and skewness) present an interpretation of the signal root, biasing effects, and noise suppression capability of morphological filters.**

### I. Introduction

Signal analysis and image processing based on mathematical morphology have been actively researched in recent years. In mathematical morphology, a signal is operated on by several morphological operations using another signal—often referred to as the structuring element—which is simpler in nature than the signal under study. For example, by selecting a suitable structuring element, morphological operations can be used for signal smoothing and noise suppression. A general morphological filter consists of sequential morphological operators such as dilation, closing, clos-erosion, and clos-opening. It is important to point out that the order of the operations is critical because different results are obtained, depending on which operation is done first.

To understand the performance of sequential morphological operations and capitalize on their applications for noise suppression, the statistical properties of morphological operations have been investigated [1]–[3]. Stevenson and Arce [1] derived an expression for the output distribution function of the open-closing and clos-opening for a class of morphological filters for which threshold decomposition is applicable. Koskinen *et al.* [2], [3] derived an expression for the output distribution function of opening, closing, and clos-opening, utilizing the relationship between morphological and stack filters. To combat analytical complexity, these studies have been restricted to flat structuring elements.

The importance of using different structuring elements for morphological operations has been reported in the past (for example, see [4]–[6]). A nonconstant structuring element weights the signal samples within the observation window differently, and the effect of noise suppression of morphological processors is governed not only by the width but also by the shape and amplitude distribution of the structuring element. In the field of detection (radar, sonar, ultrasound, etc.), noise has different distribution functions, and in this study, the performance of morphological filters for three frequently recognized noise density functions (uniform, Gaussian, and Rayleigh distributions) have been studied. To extend the study of the morphological operations to any shape of structuring element and other sequential stages such as clos-erosion and clos-opening, the Monte Carlo technique was used to estimate the processed signal density function.

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### II. Dilation, Erosion and Closing Density Functions

Morphological operations can be applied to binary or multilevel signals. Since the binary signal is a subset of a multilevel signal, we present a brief discussion of the multilevel morphological operators that will be used throughout this paper. The multilevel dilation of two functions is defined as the top surface of dilation of their umbras. From this definition, multilevel dilation can be computed in terms of a maximum operation and a set of addition operations

$$(f \oplus s)(m) = \mathrm{MAX}_{n=m-M+1,\cdots,m}[f(n) + s(m-n)] \quad \text{for} \quad m = M-1, M, \cdots, N-1 \tag{1}$$

where $f$ and $s$ are two discrete functions defined on $F = \{0, 1, \cdots, N-1\}$ and $S = \{0, 1, \cdots, M-1\}$, respectively. It is assumed that $N > M$.

The definition of multilevel erosion is similar to the definition of multilevel dilation. The multilevel erosion of the function $f$ by a multilevel structuring element $s$ is the top surface of the binary erosion of the umbra of the function $f$ by the umbra of the structuring element $s$. Multilevel erosion can be evaluated in terms of a minimum operation and a set of subtraction operations

$$(f \Theta s)(m) = \mathrm{MIN}_{n=0,\cdots,M-1}[f(m+n) - s(n)] \quad \text{for} \quad m = 0, \cdots, N-M. \tag{2}$$

The multilevel opening of $f$, by structuring element $s$, is defined as

$$f \circ s = (f \Theta s) \oplus s. \tag{3}$$

The multilevel closing of $f$, by structuring element $s$, is

$$f \bullet s = (f \oplus s) \Theta s. \tag{4}$$

The dilation of the function $f$ by a structuring element $s$ can be related to order statistic (OS) filters. By combining the definition of OS with the definition of dilation, the $M$th OS of any signal by window $M$ coincides with dilation by a flat structuring element with a width $M$ and a constant value zero. Let us assume that $X_1, X_2, \cdots, X_M$ are $M$ independent and identically distributed (iid) variates, each with a cumulative distribution function $P(x)$. Using the above relation, the dilation density function $p_d(x)$ by a flat structuring element with a width $M$ and a constant value zero becomes

$$p_d(x) = M P^{M-1}(x) p(x) \tag{5}$$

where $p(x)$ is the density function of the input $X_i$. If a flat structuring element $s$ has a width $M$ and a constant height $A$, then the dilation density function is equal to

$$p_d(x) = M P^{M-1}(x - A) p(x - A). \tag{6}$$

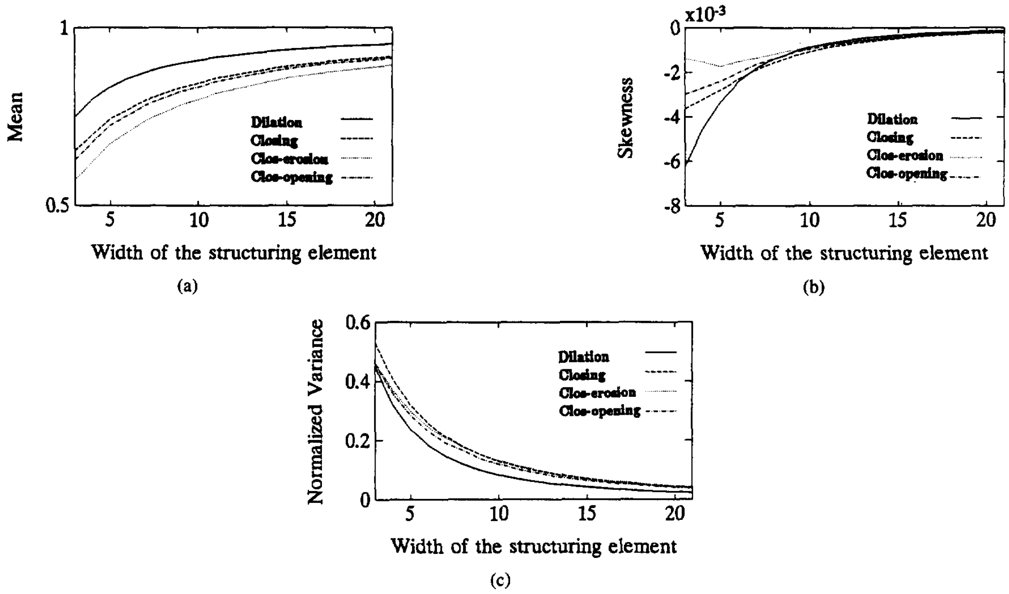


Fig. 1. Mean, normalized variance and skewness of dilation, closing, clos-erosion, and clos-opening when the input signal has a uniform density function and a flat structuring element is used.

Calculation of the dilation density function for iid inputs can be generalized to any structuring element by modifying the density function of the variates of the input signal. The modification is performed by shifting each density function to the right by a value depending on the values of the structuring element. Therefore, the dilation density function can be calculated using the statistical property of the $M$th OS density function for independent and nonidentically distributed variates [7].

$$p_d(x) = \left[\prod_{i=1}^{M} p_i(x)\right] \sum_{i=1}^{M} \frac{p_i(x)}{P_i(x)} \tag{7}$$

where $p_i(x)$ and $P_i(x)$ are the density function and distribution function of input signals modified by the values of the structuring element.

The expression for the erosion density function can be derived using a similar method applied to the derivation of the dilation density function. If the structuring element is flat with a width $M$ and a constant value $A$, then erosion coincides with the first OS of the signal inside the window $M$. Consequently, the erosion density function $p_e(x)$ can be given by

$$p_e(x) = M[1 - P(x + A)]^{M-1} p(x + A). \tag{8}$$

Calculation of the erosion density function for iid inputs can be generalized to any structuring element by modifying the density function of the input signal by a value depending on the values of the structuring element. Then, the erosion density function can be calculated using the statistical property of the first OS of independent and nonidentically distributed variates [7]

$$p_e(x) = \frac{1}{(M-1)!} \times \begin{vmatrix} p_1(x) & p_2(x) & \cdot & p_M(x) \\ 1 - P_1(x) & 1 - P_2(x) & \cdot & 1 - P_M(x) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 - P_1(x) & 1 - P_2(x) & \cdot & 1 - P_M(x) \end{vmatrix} \begin{matrix} \\ M-1 \\ \text{rows} \end{matrix} \tag{9}$$

where $|\ |$ is the permanent of a square matrix and is defined like the determinant of the matrix, except that all signs are positive [7]. The terms $p_i(x)$ and $P_i(x)$ for $i = 1, 2, \cdots, M$ correspond to the modified density and distribution functions due to the effect of values of the structuring element.

Numerical evaluation [5] of (7) and (9) using a symmetrical triangular structuring element with a width $M$ and height $A$ reveals that the noise variance of dilation or erosion reduces by increasing the width of the structuring element $M$. For the same width of structuring element, the difference between dilation or erosion density functions and the input density function becomes less significant by increasing the height of the structuring element $A$. This implies that the effective width of the triangular structuring element that is critical in reducing the variance of the noise is inversely proportional to the height of the structuring element. Both operations—dilation and erosion—introduce a bias that is proportional to the height of the structuring element.

The statistical properties of dilation and erosion can be extended to other sequential stages. In particular, a general statistical expression for a closing density function can be derived when inputs are iid by utilizing probabilistic mappings between input and output signals

$$p_c(x) = [M^2 P(x)^{M-1} - (M^2 - 1)P^M(x)]p(x) \tag{10}$$

where $P(x)$ and $p(x)$ are the distribution and density functions of the input signal, and $M$ is width of a zero value flat structuring element. The above expression is restricted to flat structuring elements since the main tool of analysis is threshold decomposition [1], [5].

To investigate the effect of $M$ on the closing density function, the mean ($E[\cdot]$), variance ($\sigma^2[\cdot]$), and skewness ($\mathrm{SK}[\cdot]$) have been evaluated. For iid inputs with uniform density functions bounded between zero and one, the mean, variance, and skewness of the closing density function $p_{c,u}(x)$ are

$$E[p_{c,u}(x)] = \frac{(M^2 + M + 1)}{(M+1)(M+2)} \tag{11}$$

$$\sigma^2(p_{c,u}(x)) = \frac{2M^3 + 4M^2 + 5M + 1}{(M+1)^2(M+2)^2(M+3)} \tag{12}$$

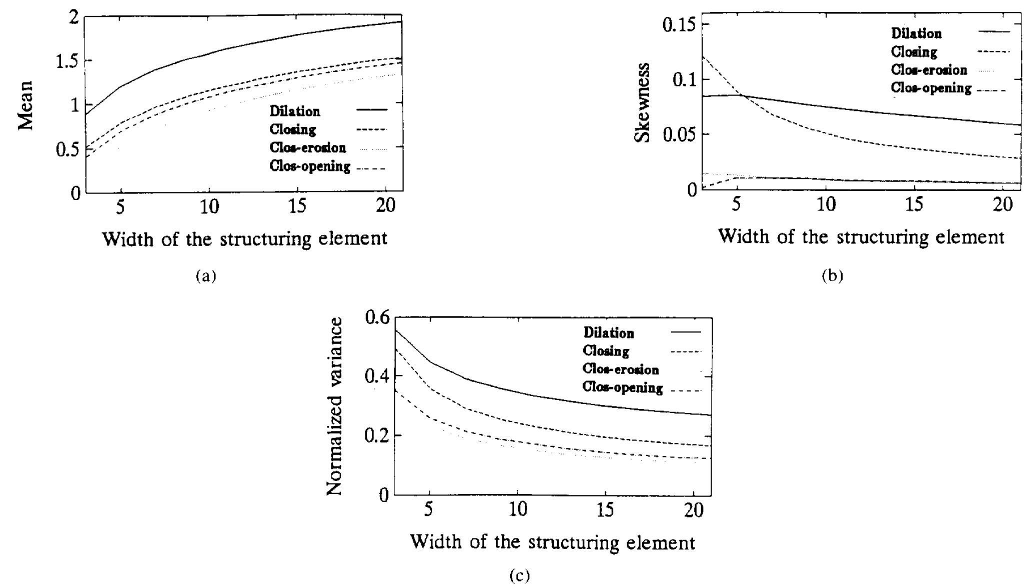


Fig. 2. Mean, normalized variance, and skewness of dilation, closing, clos-erosion, and clos-opening when the input signal has a Gaussian density function and a flat structuring element is used.

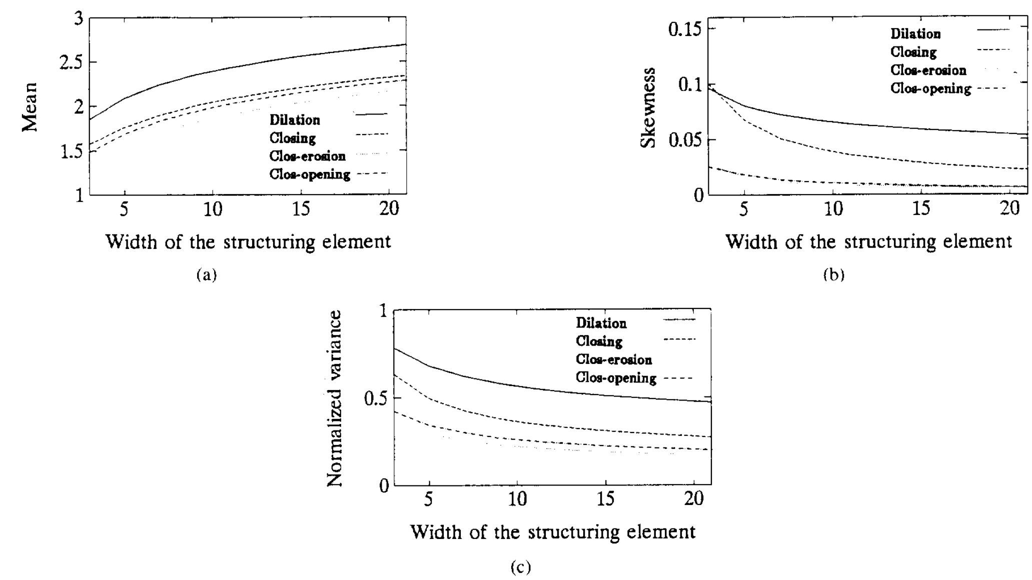


Fig. 3. Mean, normalized variance, and skewness of dilation, closing, clos-erosion, and clos-opening when the input signal has a Rayleigh density function and a flat structuring element is used.

$$\begin{aligned}\mathrm{SK}[p_{c,a}(x)] = {} & \frac{(M^2+M+3)}{(M+3)(M+4)} \\ & -3\frac{(M^2+M+2)(M^2+M+1)}{(M+1)(M+2)^2(M+3)} \\ & +2\frac{(M^2+M+1)}{(M+1)^3(M+2)^3}. \end{aligned} \tag{13}$$

For a Rayleigh density function (i.e., $p(x) = xe^{x^2/2}$ for $0 \le x \le \infty$), the mean, variance, and skewness of the closing density function $p_{c,a}(x)$ are

$$\begin{aligned}E[p_{c,r}(x)] = {} & \sqrt{\pi/2} + \left[\sum_{n=1}^{M-1}(-1)^n\frac{(M)!}{(M-n)!(n)!}\right. \\ & \left.\times \frac{(1-nM)}{(n+1)}\sqrt{\frac{\pi}{2(n+1)}}\right] \\ & -(-1)^M(M-1)\sqrt{\frac{\pi}{2(M+1)}} \end{aligned} \tag{14}$$

$$\begin{aligned}\sigma^2[p_{c,r}(x)] = {} & 2 + \left[\sum_{n=1}^{M-1}2(-1)^n\frac{(M)!}{(M-n)!(n)!}\frac{(1-nM)}{(n+1)^2}\right] \\ & -2(-1)^M\frac{(M-1)}{(M+1)} \end{aligned} \tag{15}$$

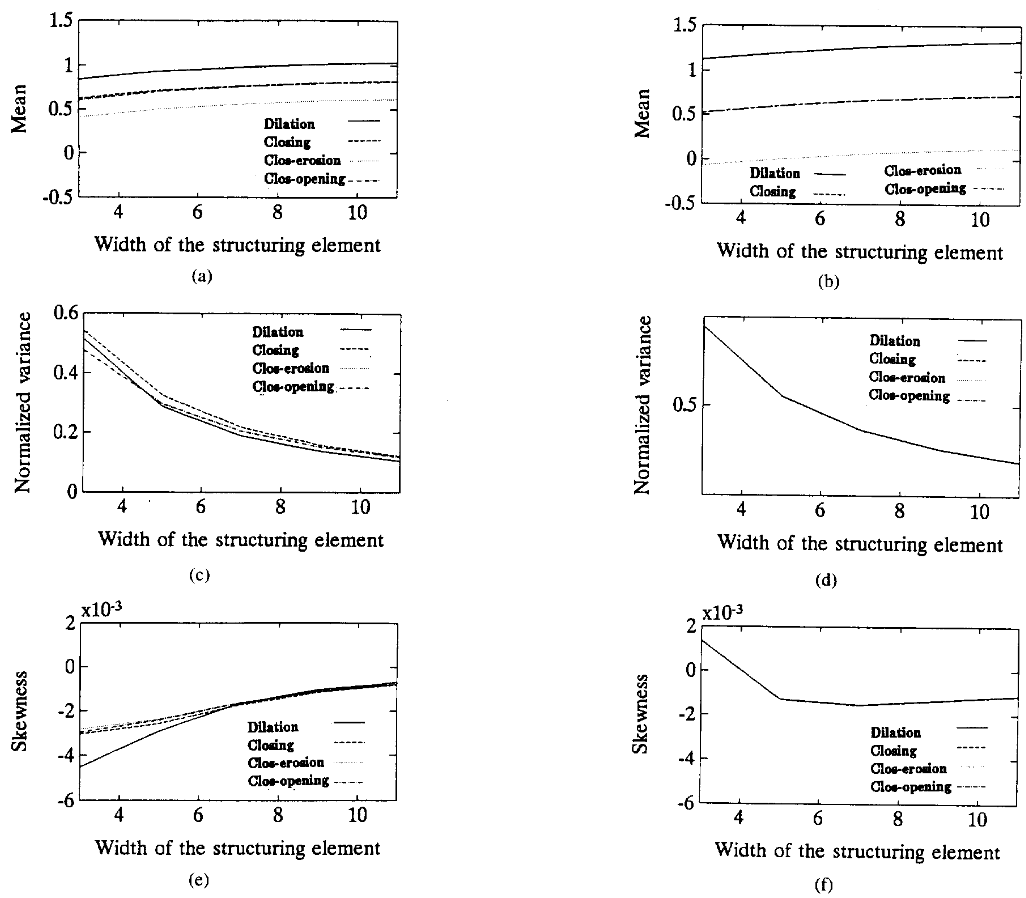


Fig. 4. Mean, normalized variance, and skewness of dilation, closing, clos-erosion, and clos-opening when the input signal has a uniform density function and a triangular structuring element with a height $A$ is used: $A = 0.2$—see (a), (c), and (e); $A = 0.6$—see (b), (d), and (f).

$$\mathrm{SK}[p_{c,r}(x)] = 3\sqrt{\pi/2} + \left[\sum_{n=1}^{M-1} 3(-1)^n \frac{(M)!}{(M-n)!(n)!} \times \frac{(1-nM)}{(n+1)^2}\sqrt{\frac{\pi}{2(n+1)}}\right] - 3(-1)^M \frac{(M-1)}{(M+1)}\sqrt{\frac{\pi}{2(M+1)}}. \quad (16)$$

For the Gaussian density function, the closed-form expression for mean, variance, and skewness cannot be obtained, although the numerical method is used [5] for evaluation.

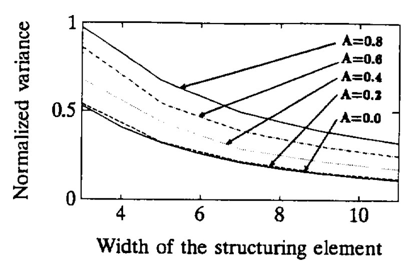


Fig. 5. Comparison between the variance of a closing density function when the input signal has a uniform density function and a flat and triangular structuring element is used.

## III. Simulation Results and Discussion

To extend the study of morphological operations to other sequential stages such as clos-erosion and clos-opening and different shapes of structuring elements, Monte Carlo simulation was used to estimate the processed signal density function when input signals have uniform, Gaussian, and Rayleigh distributions. In addition, the mean, normalized variance (i.e., the ratio of output variance over input variance) and skewness has been estimated for each of these density functions to present an interpretation of noise suppression capability of morphological filters and their biasing effects. To demonstrate the above objectives, simulation was used to process 300 input signals, each with a length of $N = 10\,000$ samples. These signals have been applied to a sequence of morphological operations, i.e., dilation, closing, clos-erosion, and clos-opening using a flat structuring element and a triangular structuring element representing a more general shape.

A comparison of the mean, normalized variance and skewness for the dilation, closing, clos-erosion, and clos-opening density functions when inputs are iid with uniform density functions (ranging between zero and one) and using flat structuring elements with different widths is shown in Fig. 1. These results indicate that dilation shifts the input signal toward the maximum values and that the signal

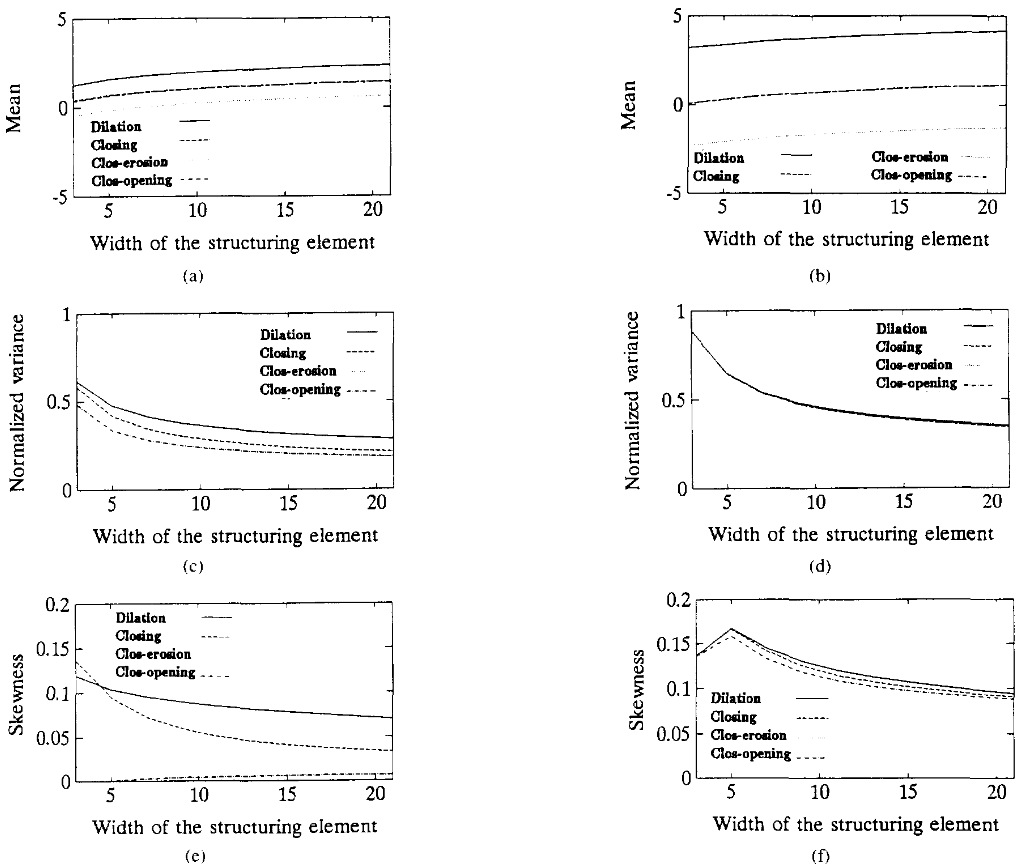


Fig. 6. Mean, normalized variance, and skewness of dilation, closing, clos-erosion, and clos-opening when the input signal has a Gaussian density function and a triangular structuring element with a height $A$ is used: $A = 0.2$—see (a), (c), and (e); $A = 0.6$—see (b), (d), and (f).

variance is reduced, resulting in a smooth operation. The closing operation (i.e., dilation followed by erosion) is a processing step toward recovering the original signal by reducing the bias caused by the dilation operation, and it slightly increases the signal variance with respect to the dilation operation. The clos-opening operation results in further smoothing of the signal, where the signal variance, mean, and skewness are less than those of the closing signal. The mean of dilation, closing, clos-erosion, and clos-opening is shifted to the right (see Fig. 1(a)) due to the effect of dilation as a first operation in the sequence of the above operations. The differences in the skewness between dilation, closing, clos-erosion, and clos-opening density functions become less significant by increasing the width of the flat structuring element (see Fig. 1(c)). Note that the height of the flat structuring element has no effect on the closing and clos-opening density function but shifts the dilation density function to the right and the clos-erosion density function to the left by a predictable value depending on the height.

Figs. 2 and 3 show the mean, variance, and skewness of dilation, closing, clos-erosion, and clos-opening density functions versus the width of the flat structuring element $M$ when inputs are iid with a Gaussian density function ($p(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$ for $-\infty \leq x \leq \infty$) or Rayleigh density function (i.e., $p(x) = xe^{x^2/2}$ for $0 \leq x \leq \infty$). Inspection of Figs. 2 and 3 suggests that the overall smoothing capability of the morphological filter applied to an input signal with a Gaussian or Rayleigh density function is similar to the results obtained for input signals with uniform distribution, although some differences can be noted. The dilation density function has the least variance when input signals are uniformly distributed, although the clos-erosion density function has the smallest variance when input signals are Gaussian and Rayleigh distributed. The differences in the variance and skewness among dilation, closing, clos-erosion, and clos-opening decrease as the width of the flat structuring element is increased, particularly when applied to input signals with uniform density functions. Contrary to this, the differences among dilation, closing, and clos-opening remain significant when the input signals have Gaussian or Rayleigh distributions. Therefore, the results describing the statistics of morphological filters applied to uniform density functions cannot be used to predict the performance associated with other types of density functions.

Consideration of a triangular structuring element (representing a general shape) in the application of morphological filters offers the option to selectively interact with the shape of the multilevel signal while reducing noise. In this study, the considered triangular structuring element is symmetrical with a width $M$ and a height $A$. Fig. 4 shows the mean, normalized variance, and skewness of dilation, closing, clos-erosion, and clos-opening using triangular structuring elements with different heights $A = 0.2$ and $0.6$ (i.e., 20 and 60% of the dynamic range of the input signal), and different widths $M$. For the same width of a triangular structuring element, the mean of the dilation density function is biased to higher values while the mean of the closing, clos-erosion, and clos-opening density functions

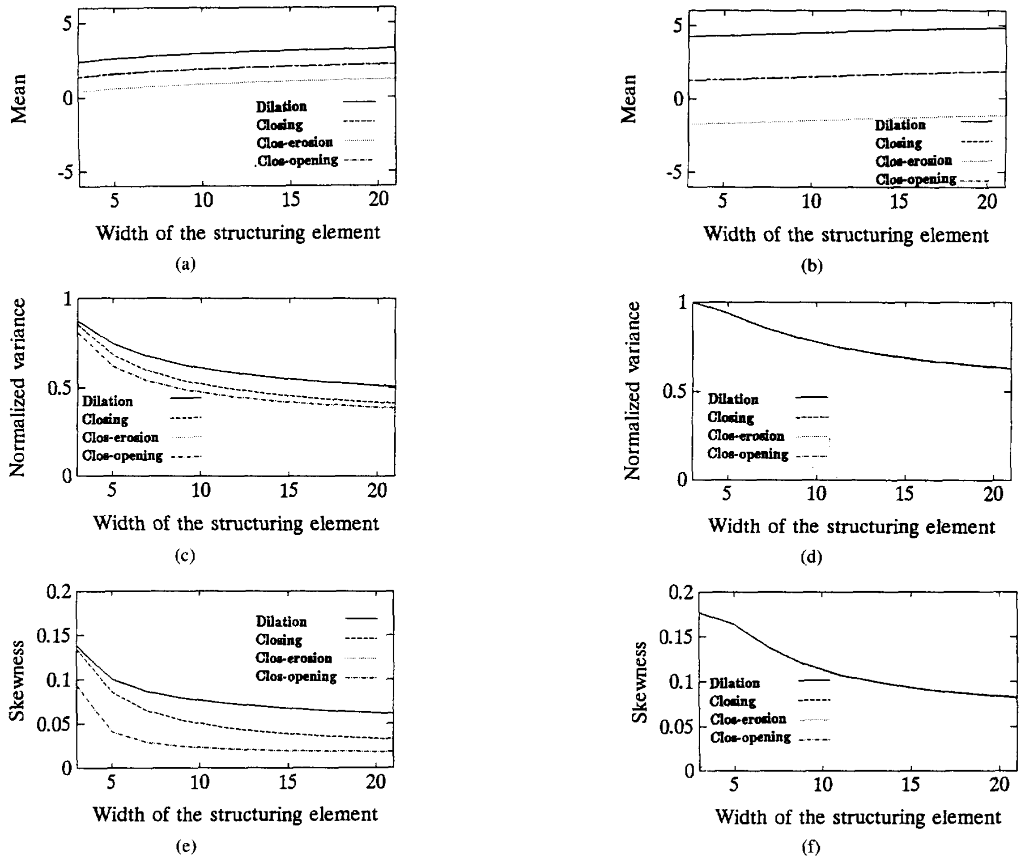


Fig. 7. Mean, normalized variance and skewness of dilation, closing, clos-erosion and clos-opening when the input signal has a Rayleigh density function and a triangular structuring element with a height $A$ is used: $A = 0.2$—see (a), (c), and (e); $A = 0.6$—see (b), (d), and (f).

decreases by increasing the height. For a given height $A$, the mean of the dilation, closing, clos-erosion, and clos-opening density functions increases by increasing the width of the triangular structuring element, and the variance decreases. In summary, the smoothing characteristics of morphological filters become more pronounced by increasing the width or decreasing the height of the triangular structuring element.

The closing density function has a slightly greater mean, variance, and skewness than the clos-opening density function when $A = 0.2$ (i.e., $A$ is 20% of the dynamic range of the input signal), but their statistics are equal when $A = 0.6$ (i.e., the closing and clos-opening have the same density functions). In addition, the clos-erosion density function is equal to the closing density function shifted by 0.6. This means that the root of the signal is found after the first stage (i.e., dilation) when the input has a uniform density function and the height of the triangular structuring element $A = 0.6$. Therefore, if operations are repeated after dilation, only the density function is shifted to right or left, depending on the operation (dilation or erosion).

A comparison between the variance of closing density functions (input has uniform density function) using a flat structuring element ($A = 0$) and a triangular structuring element with different heights ($A > 0$) is shown in Fig. 5. Inspection of these results indicate that to obtain the same variance equal to 0.03, a flat structuring element with a width of $M = 5$, a triangular structuring element with a width of $M = 7$, and a height of $A = 0.4$ or a triangular structuring element with a width of $M = 9$ and a height of $A = 0.6$, can be used. This suggests that to obtain the same variance, one can use a flat structuring element with a smaller width than a triangular structuring element.

The statistical analysis of the triangular structuring element has also been extended to iid input signals with Gaussian and Rayleigh distributions. The mean, variance, and skewness of dilation, closing, clos-erosion, and clos-opening output density functions are shown in Figs. 6 and 7 for different widths $M$ and heights $A$ of triangular structuring elements. These results show that the statistical outcome of morphological filters using a triangular structuring element is similar to that of a flat structuring element with a few noticeable differences. For triangular structuring elements having the same height, the dilation, closing, clos-erosion, and clos-opening density functions become less skewed by increasing the width ($M \geq 5$) when the inputs are Gaussian or Rayleigh distributed. However, when the inputs are uniformly distributed, the changes in skewness are irregular and not predictable (for clarification, see Fig. 4).

## IV. Conclusion

In this paper, a mathematical expression for the dilation and erosion density functions of iid inputs is derived for any structuring element using the statistical properties of order statistic filters. In addition, a statistical expression for the closing density function is derived when inputs are iid for any shape density function and a flat structuring element since the main tool of analysis is threshold decomposition. Furthermore, the statistical properties of sequential morphological operations (i.e., dilation, closing, clos-erosion, and clos-opening) are examined using different shapes of structuring elements applied to signals with uniform, Gaussian, and Rayleigh density functions. The effect of the parameters of the structuring elements are evaluated using Monte Carlo simulation. The simulated results indicate that by increasing the width of the flat structuring element, the mean (bias) becomes larger, and the variance and skewness of dilation, closing, clos-erosion, and clos-opening density functions become smaller. In addition, dilation is the most effective step in the smoothing operation. The statistical effect of all operations following dilation is rather small, especially when the width of the structuring element is large. For a uniform density function and certain height of the triangular structuring element, the dilated signal is a root signal, i.e., becoming invariant to any proceeding morphological operations using the same structuring element. For structuring elements with the same width, the flat structuring element has better noise suppression (less variance) than the triangular structuring element, although a triangular structuring element similar to certain patterns of the input signal may be desirable in preserving these patterns.

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# Stable Computation of the Complex Roots of Unity

Stephen R. Tate

*Abstract*—**In this correspondence, we show that the problem of computing the complex roots of unity is not as simple as it seems at first. In particular, the formulas given in a standard programmer's reference book (Knuth, *Seminumerical Algorithms*, 1981) are shown to be numerically unstable, giving unacceptably large error for moderate sized sequences. We give alternative formulas, which we show to be superior both by analysis and experiment.**

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## I. Introduction

In most efficient implementations of the fast Fourier transform (FFT), tables of the powers of the roots of unity are precomputed so that expensive trigonometric function evaluation can be avoided when computing the transform. In this correspondence, we consider the problem of precomputing these values. For a good survey of issues arising in computing the FFT, the reader may refer to the excellent paper by Duhamel and Vetterli [1]. In particular, we show that the method given in a standard reference text [2] is numerically unstable, and can produce very inaccurate values for moderately sized sequences. More importantly, we present an alternative way of calculating the roots of unity, present analysis that shows its superiority over the previous method, and finally give empirical results showing that the benefits of the new method are indeed substantial. In addition, while computation time is not a big issue in the computations that we describe, our methods are computationally simpler than the previous method, so there seems to be no reason to choose the previous method over our new method.

We consider the problem of computing the complex roots of unity $\omega_n^k = e^{2\pi ik/n}$ for $k = 0, 1, \cdots, n-1$, where $n$ is a power of two. It is well known that if the set of values $W_r = e^{2\pi i/2^r}$ are known for $r = 0, 1, \cdots, \log_2 n$, then the entire table of $\omega_n^k$ values can be computed by a very simple linear time algorithm (only a single complex multiplication is required per table entry). The accuracy of the $\omega_n^k$ values depends entirely on the accuracy of the $W_r$ values, so we concentrate on computing the $W_r$ values.

We can consider computing the real and imaginary parts of $W_r = c_r + is_r$ separately, where

$$c_r = \cos\left(\frac{2\pi}{2^r}\right) \quad s_r = \sin\left(\frac{2\pi}{2^r}\right). \tag{1}$$

In a standard programmer's reference book [2, p. 292], Knuth gives the following recurrence equations for computing these sequences:

$$\tilde{c}_{r+1} = \sqrt{\frac{1+\tilde{c}_r}{2}} \quad \tilde{s}_{r+1} = \sqrt{\frac{1-\tilde{c}_r}{2}}. \tag{2}$$

While the equation for $\tilde{c}_r$ is stable, it should be obvious that since $\tilde{c}_r$ approaches one rather quickly, the equation for $s_r$ involves the subtraction of almost equal quantities and thus is very unstable.

In this correspondence, we propose the following set of equations:

$$\tilde{c}_{r+1} = \frac{1}{2}\sqrt{2(1+\tilde{c}_r)} \quad \tilde{s}_{r+1} = \frac{\tilde{s}_r}{\sqrt{2(1+\tilde{c}_r)}}. \tag{3}$$

We will show by both analysis and experiment that the equations in (3) have much better error properties than the equations in (2). Furthermore, notice that in the equations in (3) a common factor has been found for both equations, meaning that only one square root needs to be taken when evaluating *both* equations. Therefore, not only are the new equations better in terms of error propagation, but they are also computationally simpler.

## II. Analysis

In this section, we will show that the formulas in equation (3) are correct mathematical representations of the series defined in equation (1), and that the formulas in equation (3) are stable.

The correctness of the formulas in (3) is obvious from the standard double-angle identities of trigonometry (see, for example, [3]), namely

$$\cos\alpha = \sqrt{\frac{1+\cos 2\alpha}{2}} \quad \sin\alpha = \frac{\sin 2\alpha}{2\cos\alpha}.$$

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