Performance Analysis of Linearly Combined Order Statistic CFAR Detectors

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Linearly combined order statistic (LCOS) constant false-alarm rate (CFAR) detectors are examined for efficient and robust threshold estimation applied to exponentially distributed background observations for improved detection. Two optimization philosophies have been employed to determine the weighting coefficients of the order statistics. The first method optimizes the coefficients to obtain efficient estimates of clutter referred to the censored maximum likelihood (CML) and best linear unbiased (BLU) CFAR detectors. The second optimization involves maximizing the probability of detection under Swerling II targets and is referred to as the most powerful linear (MPL) CFAR detector. The BLU-CFAR detector assumes no knowledge of the target distribution in contrast to the MPL-CFAR detector which requires partial knowledge of the target distribution. The design of these CFAR detectors and the probability of detection performance are mathematically analyzed for background observations having homogeneous and heterogeneous distributions wherein the trade-offs between robustness and detection performance are illustrated.

Manuscript received September 9, 1991; revised February 11, 1993.

IEEE Log No. T-AES/31/2/08018.

This work is supported by a grant from SDIO/IST and managed by the Office of Naval Research under Contract N00014-86-K-0520.

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I. INTRODUCTION

In radar systems, interfering signals from the environment can degrade the detection of received echoes from a target within a particular radar cell. Often *a priori* knowledge of the parameter(s) of the clutter distribution at any given time are not available, thus, the performance of optimally chosen (i.e., Neyman–Pearson) fixed-threshold CFAR detectors deteriorates significantly. In such instances, the need arises for a nonparametric or constant false-alarm rate (CFAR) detector which is designed to be insensitive to changes in the density functions of the clutter. CFAR detectors have been utilized in radar systems where the clutter environment is partially unknown and/or has varying statistical properties (e.g., power, etc.). An effective method of compensating for changes in the clutter statistics is to use local threshold estimates from background clutter observations.

In the past, CFAR detectors have been implemented using observations in surrounding radar cells to create local estimates of the threshold when these observations contain predominant clutter information [1]. A schematic of the CFAR detection system considered here is shown in Fig. 1. A single test observation y is classified to belong either to the null hypothesis (clutter) H_0 or the alternative hypothesis (target-plus-clutter), H_1, by using a local threshold $\hat{T}$, estimated from a set of assumed clutter observations (belonging to H_0), $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$. The amplitude of the clutter observations $\mathbf{x}$ is randomly distributed, and for this study is assumed to belong to the exponential distribution corresponding to the output of a square law detector where the clutter is Rayleigh distributed [2]:

$$F_0(x) = 1 - \exp(-x/\mu) \tag{1}$$

where μ is the unknown scale parameter (i.e., related to the power).

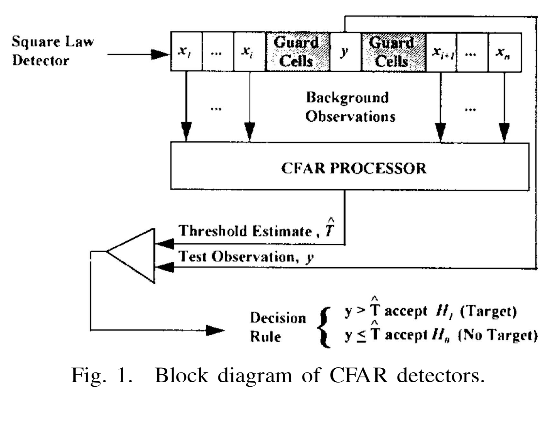


Fig. 1. Block diagram of CFAR detectors.

It has been shown that the cell averaging (CA) CFAR detector (i.e., $\hat{T} \propto \sum_{i=1}^{n} x_i$) performs optimally (offers maximum probability of detection) for homogeneous and exponentially distributed clutter observations and Swerling II target models. The

CA-CFAR detector is based on an application of the *invariance principle* [3–5] in the estimation of the scale parameter μ. However, the performance of this optimized detector can significantly deteriorate when the assumption of homogeneous observations is violated (e.g., the introduction of target information and/or spurious noise). Thus, alternative methods that censor undesirable information (outliers) from the threshold estimate are required to make the detector perform robustly.

The performance of the CFAR detector depends on the validity of assumed clutter observations $\mathbf{x}$. Potential target information contained in the surrounding cells of $\mathbf{x}$ will generally degrade the probability of detection while maintaining the CFAR constraint. However, CFAR performance may be degraded by sharp transitions in the clutter power which is dependent on the size of $\mathbf{x}, n$. Taking these two scenarios into consideration, an intuitive solution can be obtained by censoring the clutter observations from large and small deviations (i.e., outliers) which would lessen the effects of the outliers and, consequently, improve the performance of the detector. The censoring of data can be accomplished by using order statistics.

Order statistics characterize amplitude information by ranking observations in which differently ranked outputs can estimate different statistical properties of the distribution from which they stem. The order statistic corresponding to a rank r is found by taking the set of n observations $x_1, x_2, \ldots, x_n$ and ordering them with respect to increasing magnitude

$$x_{(1)} < x_{(2)} \leq \cdots x_{(r-1)} \leq x_{(r)} \leq x_{(r+1)} \cdots \leq x_{(n)} \quad \text{for} \quad 1 \leq r \leq n \tag{2}$$

where $x_{(r)}$ is the rth-order statistic.

In recent studies, censoring techniques have been implemented using order statistic (OS) CFAR detectors [6] and trimmed-mean (TM) CFAR detectors [7, 8] for exponentially distributed background observations. In utilizing censored data, the efficiency of the threshold estimate decreases with an increase in the number of censored samples. We present the analysis of linearly combined order statistics (LCOS) leading to a more efficient estimation of the threshold for OS-based CFAR detectors resulting in improved detection performance while censoring outliers.

In terms of determining the LCOS threshold estimate, two optimization methods are utilized. The first method employs efficient scale parameter estimates corresponding to the censored maximum likelihood (CML) and best linear unbiased (BLU) estimates. The CML estimate of the scale parameter provides for one-sided censoring (with respect to larger samples) whereas the BLU estimate of the scale parameter provides for two-sided censoring. Utilizing the two estimates for required censoring needs, CFAR threshold estimates can be designed which result in improved detection over OS-CFAR and TM-CFAR detectors for homogeneous background observations, $\mathbf{x}$.

Other optimization procedures involve determining the weighting coefficients, based on maximizing the probability of detection (MPL-CFAR) (MPL is most powerful linear) while assuming Swerling II targets. This method directly follows BLU estimate analysis for exponentially distributed observations where th eproblem becomes one of determining a single unknown weighting coefficient. This technique shows a slight improvement in performance over the BLU-CFAR detector, although the BLU, CML, and MPL are shown to be equivalent for one-sided censoring [12, 14]. In addition to the above performance analysis, heterogeneous background observations are used to study the effects of contaminating target information.

II. CFAR THRESHOLD ESTIMATION

The ideal CFAR threshold T for the one-sample test is given by

$$T = F_0^{-1}(1-\alpha) \tag{3}$$

where $F_0^{-1}(\cdot)$ is the inverse distribution function for the null hypothesis and α is the constant probability of a false alarm parameter of the CFAR detector. Since $F_0^{-1}(\cdot)$ is not completely known *a priori*, the null observations, $\mathbf{x}$ are used to estimate the threshold $\hat{T}$ as shown in Fig. 1. The observations are assumed to be homogeneous, stemming from the same distribution. Then, the actual threshold can be considered a nonrandom quantity, and the estimation of the threshold can be thought of as a point estimation problem.

The probability of false alarm for the CFAR detector P_{FA} is given by

$$P_{\mathrm{FA}} = \Pr\{Y > \hat{T} \mid H_0\} \tag{4}$$

where Y is the random variable corresponding to the test observation with a distribution function $F_0(x)$ and a density function $f_0(x)$ under the null hypothesis H_0. Note that $\hat{T}$ is a random variable with a distribution function $F_{\hat{T}}(\tau)$, a density function $f_{\hat{T}}(\tau)$, and observed values of $\hat{T}$ denoted by τ. P_{FA} can be expressed as

$$P_{\mathrm{FA}} = \int_0^{\infty} [1 - F_0(\tau)] f_{\hat{T}}(\tau)\, d\tau \tag{5}$$

under the assumption that the random variables $\hat{T}$ and Y are independent.

The above expression shows the effect of the density function of the threshold estimate, $f_{\hat{T}}(\tau)$, on P_{FA}. If the estimate is asymptotically unbiased and consistent, $f_{\hat{T}}$ should converge to $\delta(\tau - F_0^{-1}(1-\alpha))$

as n approaches infinity. However, for finite samples there will generally be spreading about T governed by $f_{\hat{T}}(\tau)$. In order to satisfy the CFAR constraint described by (5), the threshold estimate must be adjusted, resulting in a biased estimate which may lower the probability of detection.

The design of the CFAR detector can be accomplished using an *invariant test* where a uniformly most powerful (UMP) invariant test can sometimes be derived for homogeneous background observations (i.e., CA for Rayleigh clutter) [3–5]. However, in instances where clutter observations are not homogeneous (i.e., containing outliers) OS and TM processors have been shown to perform robustly (in terms of CFAR loss [6, 8]). This threshold estimate is based on the invariant translations of the unknown scaling parameter, μ, given by

$$\hat{T} = \theta g(\chi) \tag{6}$$

where $\chi = \{x_{(r+1)}, x_{(r+2)}, \ldots, x_{(n-s)}\}$, θ is the design parameter to satisfy the size of the detector (i.e., probability of false alarm), and r and s are the number of smallest and largest censored observations, respectively. The function $g(\chi)$ must be chosen so that it will map the translation to each of the arguments, i.e.,

$$\theta g(\chi) = g(\theta\chi). \tag{7}$$

The above relations facilitate setting probability of false alarm (see (5)) independent of the scale parameter μ of (1) [4, 5]. For example, CA-CFAR has $g(\mathbf{x}) = \sum_{i=1}^{n} x_i$. For this study, we consider linear combinations of censored ordered observations for $g(\chi)$ and the resulting threshold is given by

$$\hat{T} = \theta \sum_{i=r+1}^{n-s} \lambda_i x_{(i)} \tag{8}$$

where λ_i are the weighting coefficients and r and s are the number of smallest and largest censored observations, respectively. Justification for the study of these linear estimators is attributed to satisfying the invariance constraint of (7) and producing tractable analytic designs. In addition, one-sided censoring (i.e., $r = 0$) provides optimal performance (i.e., Swerling II targets) for the reduced sample size $n - s$ with homogeneous background observations, $\mathbf{x}$ [3–5].

A threshold estimate $\hat{T}$ can be written in terms of estimates for the unknown scale parameter, μ [8, 9]:

$$\hat{T}(\chi) = \theta\hat{\mu}(\chi) \tag{9}$$

where $\hat{\mu}$ is the estimate of the scale parameter of the clutter distribution and θ is a design parameter which is set in accordance to the false alarm constraint (5). From here, there are several approaches to optimizing the weighting coefficients λ_i. Next, the BLU and the MPL are examined in situations where the varying degrees of clutter information are available. These optimization techniques lead to the determination of the weighting coefficients λ_i and the design parameter θ, satisfying the CFAR constraint.

III. LCOS–CFAR DETECTORS

Estimation of the CFAR threshold for clutter distributions with an unknown scale parameter μ can be improved by employing the sum of weighted order statistics in an efficient manner. Based on unknown values of μ, from the expression in (9), it is clear that the lower the variance of the unbiased estimate $\hat{\mu}$, the closer the threshold estimates will be to the optimal value T. This corresponds to the BLU estimate where it has minimum variance over all possible unbiased estimates consisting of LCOSs [9, 10]. Below, a presentation of the BLU scale parameter estimate is given in terms of multiple-order statistics and is then applied towards CFAR detection.

The BLU estimate for the scale parameter μ is given by $\hat{\mu} = \sum_{i=r+1}^{n-s} \lambda_i x_{(i)}$ where λ_i are the weighting coefficients yet to be determined. The BLU coefficients are determined by minimizing the variance of the estimate, $E[(\hat{\mu} - \mu)^2]$, subject to the unbias constraint, $E[\hat{\mu}] = \mu$. The general solution for the BLU coefficients given a univariate distributions has been derived by Lloyd [9].

Normalizing the random variable with respect to the actual value of the scale parameter μ,

$$w_{(i)} = X_{(i)}/\mu \tag{10}$$

and defining the statistics of $w_{(i)}$,

$$m_i = E[w_{(i)}] \tag{11}$$

$$\sigma_{ij}^2 = \mathrm{cov}\{w_{(i)} w_{(j)}\} = E[w_{(i)} w_{(j)}] - m_i m_j \tag{12}$$

producing quantities independent of μ. Utilizing these statistics, the BLU estimate of the scale parameter μ is (see paper by Lloyd [9] for derivation),

$$\hat{\mu} = \frac{\mathbf{m}^T \Sigma^{-1}}{\mathbf{m}^T \Sigma^{-1} \mathbf{m}} \cdot (x_{(r+1)} x_{(r+2)} \cdots x_{(n-s)})^T \tag{13}$$

where $\mathbf{m} = \{m_i\}$ for $i = r+1, r+2, \ldots, n-s$ and $\Sigma = \{\sigma_{ij}^2\}$ which is a positive definite symmetric matrix of size $n - r - s$.

For exponentially distributed observations, $m_i = \sum_{j=0}^{i-1} (n-j)^{-1}$ (see (11)) and the elements of the covariance matrix Σ (see (12)) are given by [11]:

$$E[w_{(i)}^2] = \sum_{j=0}^{i-1} (n-j)^{-2} \qquad \text{for} \quad r+1 \le i \le n-s \tag{14}$$

and

$$E[w_{(i)}w_{(j)}] = i(j-i)\binom{n}{i}\binom{n-i}{j-i}\sum_{k=0}^{i-1}\sum_{l=0}^{j-i-1} \times (-1)^{k+l}\binom{i-1}{k}\binom{j-i-1}{l} \times H(j-i+k-l, n-j+l+1) \quad \text{for} \quad r+1 \le i < j \le n-s \tag{15}$$

where $H(a,b) = (a+3b)/b^2(a+b)^3$.

The coefficients λ_i have been numerically determined, however, the accuracy of the solutions (e.g., using double precision arithmetic) deteriorates significantly for moderately large n (i.e., $n > 23$) in the case of exponential [12] and Rayleigh distributed observations [10]. This limit in numerical precision can be alleviated for exponentially distributed observations by finding the explicit solutions for $\boldsymbol{\lambda}$ (see Appendix A):

$$\lambda_i = \frac{1}{\Delta}\begin{cases} s+1 & \text{for} \quad i = n-s \\ 1 & \text{for} \quad r+2 \le i < n-s \\ -n+r+1+\dfrac{\sum_{j=0}^{r}(n-j)^{-1}}{\sum_{j=0}^{r}(n-j)^{-2}} & \text{for} \quad i = r+1 \end{cases} \tag{16}$$

where

$$\Delta = n-r-s-1+\frac{(\sum_{j=0}^{r}(n-j)^{-1})^2}{\sum_{j=0}^{r}(n-j)^{-2}}. \tag{17}$$

This solution significantly reduces the number of computations needed and substantially increases the precision of the solution for large n. Below, the CFAR design parameter θ is found explicitly for the exponential clutter distribution utilizing the BLU solution in (17). The BLU-CFAR threshold estimate is given by (8).

In order to determine the probability of false alarm in terms of the design parameter θ (see (8)), the density function of the threshold estimate is required (see (5)). To simplify the analysis, the characteristic function of the threshold is determined and used to evaluate the probability of false alarm for exponentially distributed background observations.

When the observations are exponentially distributed, the clutter variates $\mathbf{x}$ can be written in terms of independent random variables Y_i where $i = 1,\ldots,n-r-s$ [13]:

$$Y_1 = X_{(r+1)} \tag{18}$$

and

$$Y_i = X_{(i+r)} - X_{(i-1+r)} \quad \text{for} \quad 2 \le i \le n-r-s. \tag{19}$$

The joint density function of $\mathbf{Y}$ is given by [13]:

$$f_{\mathbf{Y}}(y_1,\ldots,y_{n-s-r}) = \frac{n!}{\mu^{n-r-s}r!(n-s)!}[1-e^{-y_1/\mu}]^r \times \exp\left\{-\sum_{i=1}^{n-s-r}(n-r-i+1)y_i/\mu\right\} \tag{20}$$

where the density of each independent random variable Y_i, is

$$f_{Y_1}(y_1) = \frac{(r+1)}{\mu}\binom{n}{r+1}e^{-(n-r)y_1/\mu}[1-e^{-y_1/\mu}]^r \tag{21}$$

and

$$f_{Y_i}(y_i) = \frac{(n-r-i+1)}{\mu}e^{-(n-r-i+1)y_i/\mu} \quad \text{for} \quad 2 \le i \le n-r-s. \tag{22}$$

The probability density function of $\hat{T}$ from (8) can be determined from the product of the characteristic functions of the independent random variables Y_i, $i = 1,2,\ldots,n-r-s$. The characteristic function of a random variable Y is defined as

$$\Phi_Y(\omega) = \int_0^\infty f_Y(y)e^{j\omega y}\,dy \tag{23}$$

where ω is the variable corresponding to the transformed observation space. The characteristic functions of Y_i are given by

$$\Phi_{Y_1} = (n-r)\binom{n}{r}\sum_{i=0}^{r}\frac{(-1)^{r-i}\binom{r}{i}}{n-i-j\mu\omega} \tag{24}$$

$$\Phi_{Y_i} = \left[1-\frac{j\mu\omega}{n-r-i+1}\right]^{-1} \quad \text{for} \quad 2 \le i \le n-r-s. \tag{25}$$

Rewriting the (see (8)) in terms of a linear combination of y_i (see (18)–(19)):

$$\hat{T} = \theta\sum_{i=r+1}^{n-s}\left(\lambda_i\sum_{j=1}^{i-r}y_j\right) \tag{26}$$

$$\hat{T} = \theta\sum_{i=1}^{n-s-r}\xi_i y_i \tag{27}$$

where

$$\xi_i = \sum_{j=i+r}^{n-s}\lambda_j \quad \text{for} \quad i = 1,2,\ldots,n-r-s. \tag{28}$$

The characteristic functions of these random variables are given in (24) and (25), and the corresponding characteristic function for the threshold is given by

$$\Phi_{\hat{T}}(\omega) = \prod_{i=1}^{n-r-s} \Phi_{Y_i}(\theta\xi_i\omega) \tag{29}$$

$$\Phi_{\hat{T}}(\omega) = \frac{n-r}{\xi_1}\binom{n}{r}\sum_{k=0}^{r}\frac{(-1)^{r-k}\binom{r}{k}}{(n-k)/\xi_1 - j\theta\mu\omega} \times \prod_{i=2}^{n-r-s}\frac{(n-r-i+1)/\xi_i}{(n-r-i+1)/\xi_i - j\theta\mu\omega}. \tag{30}$$

The above expression can be simplified by determining the values of the ξ_i from (16) and (28):

$$\xi_i = \frac{1}{\Delta}\begin{cases} n-r-i+1 & \text{for} \quad 2 \le i \le n-r-s \\ \dfrac{\sum_{j=0}^{r}(n-j)^{-1}}{\sum_{j=0}^{r}(n-j)^{-2}} & \text{for} \quad i = 1. \end{cases} \tag{31}$$

The characteristic function of (30) can now be written as

$$\Phi_{\hat{T}}(\omega) = \frac{n-r}{\xi_1\Delta^{-n+r+s+1}}\binom{n}{r}(\Delta - j\theta\mu\omega)^{-n+r+s+1} \times \sum_{k=0}^{r}\frac{(-1)^{r-k}\binom{r}{k}}{(n-k)/\xi_1 - j\theta\mu\omega}. \tag{32}$$

It can be seen for exponentially distributed background observations, that by evaluating $\Phi_{\hat{T}}(\omega)$ at $-1/j\mu$, (23) is identical to the expression for P_{FA} in (5) for exponentially distributed observations where $f_{\hat{T}}(x) = f_T(x)$ and $1 - F_0(x) = e^{-x/\mu}$ [8, 12]. Thus, P_{FA} of the BLU-CFAR detector is found to be:

$$P_{\text{FA}} = \frac{n-r}{\xi_1\Delta^{-n+r+s+1}}\binom{n}{r}(\Delta+\theta)^{-n+r+s+1} \times \sum_{k=0}^{r}\frac{(-1)^{r-k}\binom{r}{k}}{(n-k)/\xi_1 + \theta} \tag{33}$$

where ξ_1 is given in (31). The scaling factor θ corresponding to a given P_{FA} can be numerically determined from the above equation.

In the BLU-CFAR detector a special case arises when $r = 0$, for the threshold estimate takes on the form of the CML statistic [10, 13, 14, 17] resulting in an optimal performance similar to that of the CA-CFAR detector with the reduced sample size, $n - s$. The CML-CFAR threshold is given by $\hat{T} = \theta[\sum_{i=1}^{n-s-1} x_{(i)} + (s+1)x_{(n-s)}]/(n-s)$ and the probability of false alarm [12, 17]:

$$P_{\text{FA}} = \left(1 + \frac{\theta}{n-s}\right)^{-n+s}. \tag{34}$$

The use of the CML-CFAR detector has been designed specifically for one-sided censoring and is inappropriate when two-sided censoring is needed. The censoring of lower ranks can produce more robust performance when abrupt changes in the noise power occur or when erroneous small-value observations are received. Such circumstances will negatively bias the threshold and increase the probability of false alarm.

So far the presentation of the BLU-CFAR detector has been optimized in terms of the scale parameter estimate, $\hat{\mu}$ which is a most powerful test for Swerling II targets since the estimate is sufficient and complete under one-sided censoring [14]. However, if the censoring of smaller samples is required, the sufficiency and completeness properties of the estimator are not satisfied with the BLU estimate and the optimization of the λ_i can be accomplished using the knowledge of the target distribution. The MPL CFAR detector is considered where the coefficients, λ_i are determined in order to maximize the probability of detection for Swerling II targets. Next, a discussion is given to facilitate the optimization of the coefficients under exponentially distributed background observations. As was shown in the BLU estimate, the censored observations, $X_{r+1}, \ldots, X_{n-s}$, can be linearly transformed into $n-r-s$ independent variates (see (18)–(19)) and the $n-r-s-1$ of these variates $Y_2, \ldots, Y_{n-s-r}$ (see (22)) can be linearly transformed to identical distributions by a factor of $n-r-i+1$. Since these linearly transformed variates contain equivalent statistical information, they are equivalently weighted for the MPL-CFAR detector. The coefficients of the original ordered observations, χ, are then given by

$$\lambda_i = \begin{cases} s+1 & \text{for} \quad i = n-s \\ 1 & \text{for} \quad r+2 \le i < n-s \\ \lambda & \text{for} \quad i = r+1 \end{cases} \tag{35}$$

where the coefficient λ of the smallest censored observation $x_{(r+1)}$ and the design parameter θ (see (8)) have yet to be determined. The design parameter θ is found from (33) (note: $\xi_1 = n - r - 1 + \lambda$) in accordance with the level of probability of false alarm whereas the value of λ is chosen to maximize the probability of detection of the CFAR detector. In the next section, the probability of detection performance is illustrated for Swerling II targets under homogeneous and heterogeneous background observations.

In the past, several OS-based CFAR detectors have been analyzed which are specific cases of the LCOS-CFAR detectors. The OS-CFAR detector was introduced by Rohling [6] where a single scaled order statistic is used as a threshold. The second extension incorporates several order statistics which are combined linearly with equal weighting referred to as the TM CFAR detector. The OS-CFAR [6] and

TABLE I
CFAR Detector False Alarm Performance Given l Null Observations

CFAR Detectors	Design Parameter θ @ $n=25$, & $\alpha=10^{-3}$	P_{FA} calulated from θ @ $l=5$ & $n'=20$
BLU	0.413362 ($r=5$ & $s=5$)	5.5744×10^{-3} ($r'=0$ & $s'=5$)
TM	0.712970 ($r=5$ & $s=5$)	6.7821×10^{-3} ($r'=0$ & $s'=5$)
OS	5.543180 ($r=20$)	2.9534×10^{-3} ($r'=15$)
CA	0.318256	3.9811×10^{-3}

Note: Effective sample size $n' = n - l$ and censoring $r' = r - l$ (assuming $r \geq l$) and $s' = s$.

TM-CFAR [8] detector designs for Rayleigh clutter can be found in a similar manner from (5), where P_{FA} for the OS-CFAR detector (i.e., $\hat{T} = \theta x_{(r)}$) is given by [6]

$$P_{\mathrm{FA}} = \binom{n}{r} B(r, n-r+\theta+1) = \frac{n!\,\Gamma(n-r+\theta+1)}{(n-r)!\,\Gamma(n+\theta+1)} \tag{36}$$

where $B(\cdot)$ and $\Gamma(\cdot)$ are Beta and Gamma functions [15], respectively. The P_{FA} for the TM-CFAR detector (i.e., $\hat{T} = \theta \sum_{i=r+1}^{n-s} x_{(i)}$) is given by [8]

$$P_{\mathrm{FA}} = \binom{n}{r}\binom{n-r}{s} \sum_{i=0}^{r} \frac{(-1)^{r-i}\binom{r}{i}}{(n-i)/(n-r-s)+\theta} \times \prod_{k=2}^{n-r-s}\left[\frac{n-r-k+1}{n-r-s-k+1}+\theta\right]^{-1}. \tag{37}$$

The above expressions for designing CFAR detectors assume that the background observations are comprised of homogeneous clutter observations. However, if fluctuations in the clutter power exists then the probability of false alarm rate will be altered. In terms of short duration increases of clutter power, one can expect a decrease in the probability of false alarm since the threshold values will generally increase. Likewise, sharp decreases or nulls in the background observations shall increase the probability of false alarm. The latter case is more critical since it violates the CFAR constraint. The extent at which the probability of false alarm increases can be determined from the extreme case where the background observations are contaminated with zero-valued nulls (i.e., minimum observable values). The probability of false alarm level given n observations and contamination of l null observations can be calculated for the CA, OS, TM, and BLU/MPL by determining the design parameter θ and using that design value to determine the probability of false alarm under the reality of reduced sample size $n - l$. Examples of this can be seen in Table I displaying moderate increases in P_{FA} (note 20% of the background observations contain no information) where the OS and CA-CFAR detectors show least sensitivity to null observations. If this variation is unacceptable then the detector can be designed to account for the short periods of null observations, however, the probability of detection performance will be comprised.

Other factors in detection performance include the presence of large contaminating observations which significantly degrades the probability of detection performance while satisfying the CFAR constraint. Thus, to fully understand the performance trade-offs of the CFAR detectors, the probability of detection is examined when background observations are heterogeneously distributed. In the following section, OS, TM, CML, BLU, and MPL-CFAR detectors are implemented for different censoring scenarios and detection performance comparisons are made for heterogeneous and heterogeneous background observations.

IV. PROBABILITY OF DETECTION PERFORMANCE ANALYSIS OF LCOS-CFAR DETECTORS FOR HOMOGENEOUS BACKGROUND OBSERVATIONS

The focus of the previous sections has been to utilize order statistic(s) as threshold estimators in order to maintain CFAR performance of the detector. In this section the robustness of LCOS-CFAR processors is examined in terms of their power or probability of detection performance. The probability of detection for LCOS-CFAR detectors is examined for the asymptotic case and subsequently analyzed for Swerling II targets under homogeneous clutter observations.

The ideal probability of detection, P_{D} for a target-plus-clutter distribution, $F_1(x)$, is as follows:

$$P_{\mathrm{D}} = \Pr\{Y \geq T \mid H_1\} = 1 - F_1(T) \tag{38}$$

where T is the actual threshold producing the optimal Neyman–Pearson detector. Due to the unavailability of the actual threshold value, and where the threshold estimate is a function of the censored clutter observations, $x_{(r+1)}, x_{(r+2)}, \ldots, x_{(n-s)}$, the $P_{\rm D}$ corresponding to this threshold estimate can be found through similar derivations of $P_{\rm FA}$ in (4)–(5):

$$P_{\rm D} = \int_0^\infty [1 - F_1(x)] f_{\hat{T}}(x)\, dx \tag{39}$$

where $f_{\hat{T}}(x)$ is the density for the threshold statistic $\hat{T}$.

Since all CFAR detectors considered are asymptotically unbiased and consistent threshold estimators [12, 16], the asymptotic performance, $P_{\rm D}^*$, is given by substituting the asymptotic expression for $f_{\hat{T}}(x)$:

$$P_{\rm D}^* = 1 - F_1(F_0^{-1}(1-\alpha)) \tag{40}$$

which is the optimal value for the one-sample fixed threshold detector. Thus, the LCOS-CFAR detectors considered here are asymptotically optimal. Note that for finite n, some loss in detection power will occur since the threshold will be a random variable and not fixed.

Assuming the target-plus-clutter distribution follows Swerling II model, then the target-plus-clutter distribution, $F_1(x)$, is given by [5]:

$$F_1(x) = 1 - (1 - F_0(x))^{\Theta}, \qquad \text{where} \quad 0 \le \Theta \le 1. \tag{41}$$

From this expression it can be seen that smaller values of Θ will increase the probability of detection and therefore can be thought of as being inversely related to the signal-to-clutter ratio (SCR). In particular, for Rayleigh clutter distributions, $1 + \mathrm{SCR}$ is equal to $1/\Theta$. Note that the probability of detection $P_{\rm D}$ using (39) can be obtained for exponentially distributed observations by replacing the design parameter "θ" in the expressions for $P_{\rm FA}$ ((33)–(34), and (36)–(37)) with "$\theta\Theta$" [8, 12].

The performance of the MPL- and BLU-CFAR detectors for the one-sided censoring case (i.e., $r = 0$) and Swerling II targets are equivalent and provide maximum power $P_{\rm D}$ for all SCR (i.e., Θ). However, in two-sided censoring, BLU-CFAR detectors will not meet MPL-CFAR performance. Given $\alpha = 10^{-3}$, $n = 25$, $s = 15$, and $r = 5$, the coefficients, λ_{r+1}, are numerically determined for the BLU (see (16)) and MPL-CFAR detectors (see (35)) and are shown in Fig. 2(a). The subsequent loss in $P_{\rm D}$ for the BLU-CFAR detector with respect to a MPL-CFAR detector does not exceed 10^{-7} for all SCR (see Fig. 2(b)). This is expected since the optimization of the MPL differs from the BLU in the coefficient value (see Fig. 2(a)) of the lowest ranked OS, λ_{r+1}, which contributes little information to the threshold estimate. Therefore, in the following performance comparisons, the MPL and BLU are approximately the same since no significant differences exist for the two-sided censoring cases considered.

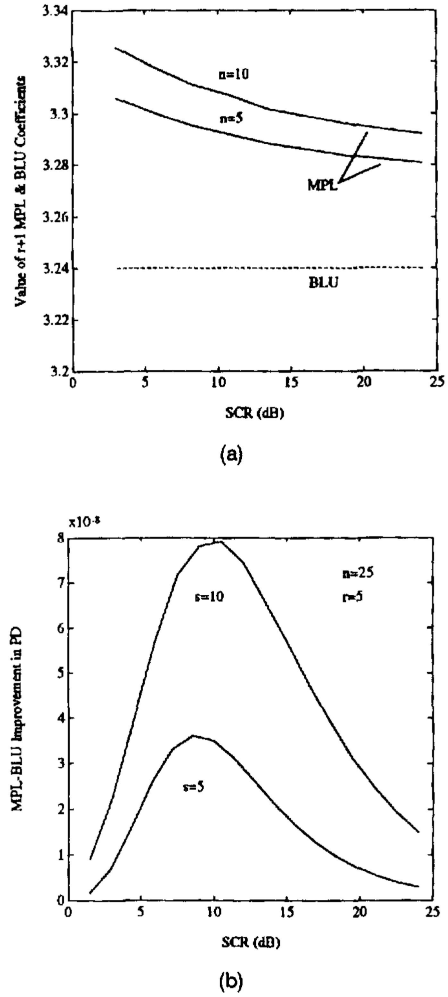


Fig. 2. (a) Values of LCOS coefficient (see (8)). λ_{r+1}, for MPL and BLU-CFAR detectors. (b) Respective improvement in performance of MPL-CFAR detector over BLU-CFAR detector with 2-sided censoring and $\alpha = 10^{-3}$.

To gauge the performance of LCOS-CFAR detectors for finite observations, the required increase in the SCR of the CFAR detector with respect to the optimal detector for the same $P_{\rm D}$ level is utilized and referred to as "CFAR Loss" [6]. An evaluation of CFAR loss is given in Fig. 3 of the BLU, CML, TM, and OS-CFAR detectors for various SCR and cases of one-censoring of homogeneous background observations. These figures illustrate the loss in efficiency of the LCOS-CFAR due to censoring possible outliers, whereas the performance of the CA-CFAR etector will have the best performance since the clutter samples are homogeneously distributed. Among the LCOS-CFAR detectors, the CML and BLU-CFAR detectors have minimum loss for all censoring cases (see Fig. 3), verifying the use of efficient parameter estimates for the threshold. However, for different degrees of censoring the improvement in performance varies. The CFAR loss in Fig. 3 for one-sided censoring shows that the improvement in performance diminishes for more

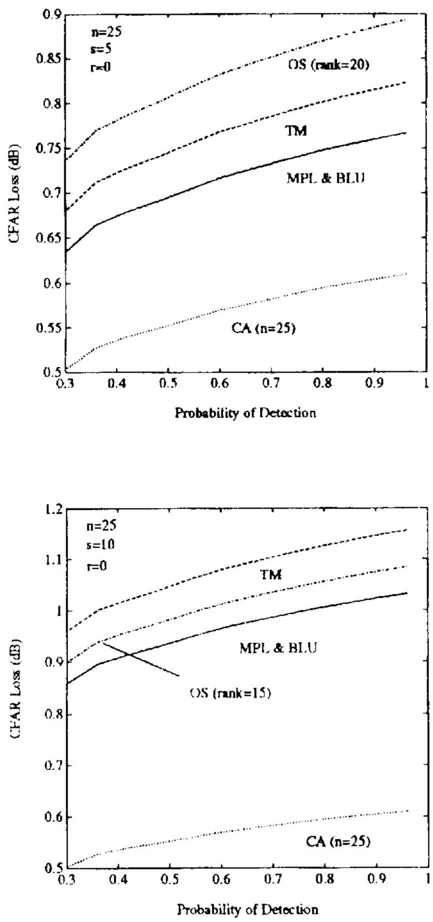


Fig. 3. CFAR loss of detectors with 1-sided censoring and $\alpha = 10^{-3}$ assuming homogeneous background observations.

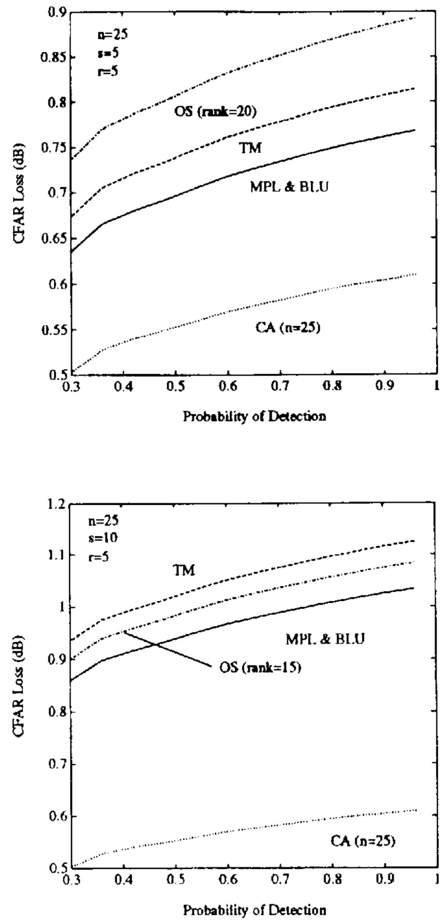


Fig. 4. CFAR loss of detectors with 2-sided censoring and $\alpha = 10^{-3}$ assuming homogeneous background observations.

censoring of the upper ranks of order statistics. Note that the OS-CFAR detector performance shown in Fig. 3 has the best performance of the ranks within the censoring requirements. It has been shown that the 80% quantile of the OS-CFAR detectors, with large n, has the optimal performance, and if additional censoring (i.e., s) is required, the largest uncensored order statistic is preferred [6, 12]. The performance of OS and TM-CFAR detectors varies for the different degrees of censoring, where, for large values of s, the TM performs less effectively than the OS threshold estimate. The reason for this is that lower order statistics have greater bias and are more inefficient estimators of the threshold which adversely affects performance when used in the TM estimate [8].

The two-sided censoring cases are shown in Fig. 4, where the BLU-CFAR detector has minimum losses of the LCOS CFAR detectors for $n = 25$. This figure also indicates the OS-CFAR detector provides improved performance over the TM-CFAR. Thus, the performance of the MPL/BLU-CFAR detector shows the best performance for homogeneous clutter observations where the censoring values r and s are chosen from *a priori* knowledge.

V. PROBABILITY OF DETECTION PERFORMANCE OF LCOS-CFAR DETECTORS FOR HETEROGENEOUS BACKGROUND OBSERVATIONS

So far, the analysis of the performance of LCOS-CFAR detectors has been made under the assumption, which may not be the case, that **x** contains only clutter information. This assumption is not critical to the P_{FA}, since the target information is assumed to be stochastically larger than the clutter which increases the expected value of the threshold estimate. However, there is a consequential loss in the probability of detection of LCOS-CFAR detectors, even in the case of extreme censoring. This loss is quantified for the LCOS-CFAR detector for Swerling II targets and several cases of censoring. Note that this case also incorporates expected loss due to sharp increases in clutter power (i.e., clutter outliers) which will have a similar effect on the detection performance.

In determining the explicit solution for P_{D} corresponding to the OS-CFAR detector (i.e., $\hat{T} = \theta x_{(r)}$), the distribution of the threshold is determined. The distribution of the order statistic has been confined to the assumption that input observations,

x, are independent and identically distributed, although it may be more appropriate to assume that input observations are independent and stem from different distributions. Under this condition, the output distribution function of rank r, $F_{X_{(r)}}(x)$, from a set of independently distributed inputs with random variables $X_1, X_2, \ldots, X_n$ and respective distributions, $F_1(x), F_2(x), \ldots, F_n(x)$ can be determined by the following relationship [13]:

$$F_{X_{(r)}}(x) = \sum_{i=r}^{n} \sum_{S_i} \prod_{l=1}^{i} F_{j_l}(x) \prod_{l=i+1}^{n} [1 - F_{j_l}(x)] \qquad (42)$$

where the summation S_i extends over all permutations $(j_1, j_2, \ldots, j_n)$ of the set of numbers, $1, 2, 3, \ldots, n$ for which $j_1 < j_2 < \cdots < j_i$ and $j_{i+1} < j_{i+2} < \cdots < j_n$. To evaluate the detection performance of the ordering operation for the inhomogeneous observations, consider the case in which the observations belong to a target which is present, H_1, and a target not present, H_0, in the background observations, **x**. Under this assumption, the probability of detection deteriorates depending on the number of target observations in **x**. A more simple expression for the output distribution function, $F_{X_{(r)}}(x)$, for the case where l observations have an F_1 distribution and $n - l$ have an F_0 distribution is given by [16]:

$$F_{X_{(r)}}(x) = \sum_{i=r}^{n} \sum_{j=\max(0,i-l)}^{\min(i,n-l)} \binom{n-l}{j} \binom{l}{i-j} F_0^j(x) \times (1 - F_0(x))^{n-l-j} F_1^{i-j}(x)(1 - F_1(x))^{l-i+j} \qquad (43)$$

analyzed in terms of target information introduced into the observations, **x** by determining P_{D} similarly to (39):

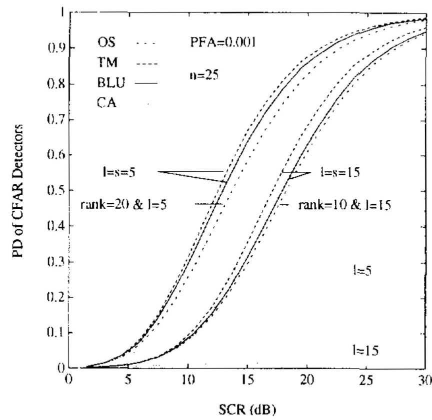
$$P_{\mathrm{D}} = \int_0^{\infty} F_{X_{(r)}}(x/\theta) f_1(x)\, dx. \qquad (44)$$

The above approach in many cases is impractical since the density function of the threshold containing several heterogeneous observations resulting in a numerical problem involving multidimensional integrals.

The detection performance of the other LCOS-CFAR detectors (e.g., TM and BLU/MPL) was evaluated utilizing a Monte Carlo generation of 10^5 pseudorandom points where the approximate P_{D} can be determined. The performance of the CA-CFAR detector for independent heterogeneous background observations can be determined by evaluating the characteristic function as follows (similar to (32)–(33)) [8]:

$$P_{\mathrm{D}} = \Phi_{\hat{T}}(-\Theta/j\mu) \qquad (45)$$

$$P_{\mathrm{D}} = \{\Phi_X(-\Theta\theta/j\mu)\}^{n-l} \cdot \{\Phi_X(-\theta/j\mu)\}^{l} \qquad (46)$$



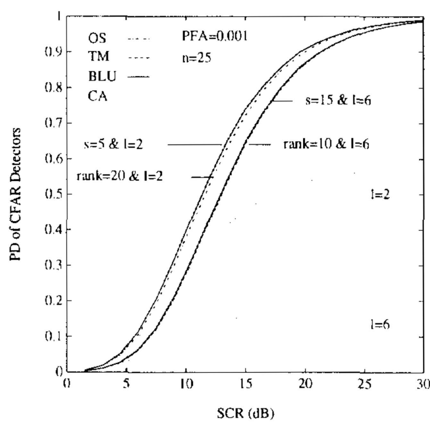


Fig. 5. Performance of CFAR detectors with one-sided censoring and $\alpha = 10^{-3}$ for heterogeneous backgroung observations (i.e., l contained observationscorrespond to target-plus-clutter information.

where $\Phi_X(\omega) = (1 - j\mu\omega)^{-1}$ for exponential distributions. The resulting probability of detection performance of the CA-CFAR detector for heterogeneous background observations containing l target observations and $n - l$ clutter observations is given by [12]:

$$P_{\mathrm{D}} = (1 + \Theta\theta)^{-n+l}(1 + \theta)^{-l}. \qquad (47)$$

Fig. 5 shows that the OS, TM, and MPL/BLU CFAR detectors perform more robustly than the CA-CFAR detector for the various cases of one-sided censoring considered. Overall, the TM-CFAR detector outperforms the MPL/BLU-CFAR detector when background observations are severely contaminated. One reason for this is due to the nature of the contaminating observations. Since the target information is assumed to be stochastically larger than the clutter (i.e., $F_1(x) < F_0(x)$, these predominant target observations effectively reduce the probability of clutter observations occurring in the censored observations corresponding to the upper ranked OSs. In turn, this reduces the actual censoring of observations belonging to the null hypothesis, whereas, the uncensored CA estimator for exponentially distributed clutter was optimal corresponding to the TM estimate with sample size $n - s$. Thus, the TM-CFAR detector will perform better in the instance where large numbers of target-plus-clutter observations exist in the **x**. The choice of whether to use one over the other in this application, modeled by Fig. 1, would depend upon the size of the target(s) with respect to the resolution of the radar cell and the given window size. If n is chosen large enough, where the contamination of the threshold estimate due to the presence of target samples was relatively small, then the BLU/MPL-CFAR detector would perform more effectively.

VI. CONCLUSIONS

In this paper we present the theoretical background and performance analysis of robust and efficient threshold estimates which have invariant CFAR performance with respect to the unknown parameters of a clutter distribution. This results in improved probability of detection of the CFAR detectors where target contamination exists, such as in the case of the CML and MPL/BLU CFAR which also exhibit efficient estimation under the homogeneous distributed background observations. Thus, the application of the MPL/BLU and CML-CFAR detectors MPL/BLU CFAR which also exhibit efficient estimation under the homogeneous distributed background observations. Thus, the application of the MPL/BLU and CML-CFAR detectors will offer improvements over the OS-CFAR and TM-CFAR in cases where the clutter has moderate contamination. The implementation of this detector (e.g., determination of r and s) still requires significant *a priori* knowledge which pertains to the size of the target with respect to the resolution of the radar cells, the occurrence of multiple targets within the null observations **x**, and the volatility of the clutter background. Assuming this information does fluctuate, the detector may be designed on the worst case scenario. Alternately, tis information can be incorporated into adaptive censoring solutions where additional hypothesis tests may be incorporated [18]. The BLU/MPL-CFAR detectors can also be utilized in the field of distributed processing [19] in which further improvements in the probability of detection can be made by utilizing the decisions of several CFAR detectors.

APPENDIX A

Statement. The solution to the BLU estimate of the scale parameter from ordered observations with a parent **exponential** distribution is

$$\hat{\mu} = \sum_{i=r+1}^{n-s} \lambda_i x_{(i)} \tag{48}$$

where the coefficients, λ_i, are given in (16).

PROOF Given that BLU estimate composed of a set of censored observations, $x_{(r+1)}, x_{(r+2)}, \ldots, x_{(n-s)}$, the coefficients can be simplified by utilizing the transformations of (18) and (19) to produce a set of independently distributed random variables, $Y_1, Y_2, \ldots, Y_{n-r-s}$. Using a second transformation of $Z_i = (n-r-i+1)Y_i$ for $i = 2, 3, \ldots, n-r-s$, the resulting random variables **Z** will be independent and identically distributed [13]. The BLU estimate utilizing **Z** is simply their sum [12], denoted by the statistic $Z_T = \sum_{i=2}^{n-r-s} Z_i$. Thus, the overall BLU estimate can be found by the linear combination of the two statistics, Y_1 and Z_T.

The overall BLU estimate is given by (13) where **m** is found using (11) [13]:

$$\mathbf{m} = [E[X_{(r+1)}]\ E[Z_T]]^T \tag{49}$$

$$\mathbf{m} = \left[\sum_{j=0}^{r} \frac{1}{n-j}\ n-r-s-1\right]^T \tag{50}$$

and the covariance matrix Σ is found from (14) [13]:

$$\Sigma = \begin{bmatrix} \sum_{j=0}^{r}(n-j)^{-2} & 0 \\ 0 & n-r-s-1 \end{bmatrix}. \tag{51}$$

Note that Z_T is the sum of independently distributed random variates with unity mean and variance, and the cross terms are zero since the two statistics are independent.

Likewise, the BLU estimate in terms of the random variables Z_T and Y_1 is derived from (13):

$$\hat{\mu} = \frac{\dfrac{\sum_{j=0}^{r}(n-j)^{-1}}{\sum_{j=0}^{r}(n-j)^{-2}} \cdot y_1 + z_T}{n-r-s-1 + \dfrac{(\sum_{j=0}^{r}(n-j)^{-1})^2}{\sum_{j=0}^{r}(n-j)^{-2}}} \tag{52}$$

translating the above expression in terms of the original observations, $x_{(r+1)}, x_{(r+2)}, \ldots, x_{(n-s)}$, by means of

$$Y_1 = X_{(r+1)} \tag{53}$$

and

$$Z_T = \sum_{i=2}^{n-r-s} (n-r-i+1)[X_{(r+i)} - X_{(r+i-1)}]. \tag{54}$$

Substituting Y_1 and Z_T into (52) provides the following BLU estimate,

$$\hat{\mu} = \frac{(s+1)x_{(n-s)} + \sum_{i=2}^{n-r-s-1} x_{(r+i)} + \left\{ \frac{\sum_{j=0}^{r}(n-j)^{-1}}{\sum_{j=0}^{r}(n-j)^{-2}} - n + r + 1 \right\} \cdot x_{(r+1)}}{n - r - s - 1 + \frac{(\sum_{j=0}^{r}(n-j)^{-1})^2}{\sum_{j=0}^{r}(n-j)^{-2}}}. \tag{55}$$

Comparing the above expression for $\hat{\mu}$ and (48), the coefficients λ_i can be identified. These coefficients agree with the BLU estimate described in (16).

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