GABOR TRANSFORM WITH OPTIMAL TIME-FREQUENCY RESOLUTION FOR ULTRASONIC APPLICATIONS

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ABSTRACT

The resolution of a t-f distribution is important for flaw detection and characterization in ultrasonic nondestructive evaluation. Wigner-Ville distribution (WVD) exhibits the true t-f resolution of an ultrasonic echo on a joint t-f plane. But this important characteristic of WVD is of no use for multiple ultrasonic echoes due to inherent cross-term artifacts. An alternative to WVD is Gabor Transform (GT), a signal decomposition method, which displays the time-frequency energy of a signal on a joint t-f plane without generating cross-terms. This desirable characteristic is achieved at the expanse of t-f resolution. In this study we are performing both analytical and numerical evaluation of GT t-f resolution for ultrasonic applications. We have generated a set of curves from the analytical solution that depict t-f resolution (concentration) of a Gaussian ultrasonic echo for different values of GT window duration. With the apriori knowledge of measurement system's characteristics we can choose an optimal GT window parameter for better detection of multiple echoes in both time and frequency. We present examples that demonstrate performance of GT with optimal resolution in comparison to WVD. In addition experimental results will be provided that demonstrate improved detection and characterization of multiple ultrasonic echoes.

I. INTRODUCTION

An effective method of time-frequency analysis of ultrasonic signals backscattered from inhomogeneous materials is to consider them as a superposition of a number of localized elementary signals (by applying a signal decomposition method). The signal decomposition method has the quality of representing the signal on a joint time-frequency plane. One of the signal decomposition methods is the constant-bandwidth analysis achieved by Gabor transform (GT). A non-trivial square integrable function $\{ g_\alpha(t) \in L^2(\Re) \}$ for time-frequency localization has finite energy [1]. The Fourier transform of a Gaussian window function, $g_\alpha(t)$, is also a Gaussian function in frequency, $G_\alpha(\omega)$, where α is a parameter related to the duration of the window function. Therefore, function $g_\alpha(t)$ can be used for t-f localization. GT window is a Gaussian pulse of constant shape modulated with a particular frequency. The width of the window function, or the width of the t-f window used for GT remains the same for localizing signals with both high and low frequencies. The Gaussian window of GT has the property of optimal concentration according to the Heisenberg's uncertainty principle and, therefore, is useful for t-f analysis. According to Heisenberg's uncertainty principle both the time and frequency spread (resolution) simultaneously cannot be made infinitely small, because of a lower bound on the time-bandwidth product,

$$\Delta t \, \Delta f \geq 1/4\pi \tag{1}$$

where Δt and Δf are defined as effective (time centroid) duration and effective frequency (frequency centroid)

The equality sign or the minimum $\Delta t \; \Delta f$ product ($= 1/4\pi$), is obtained by using Gaussian modulated pulse g (t) [2]. This minimum spread property of the modulated Gaussian pulse is used as a natural basis for the signal analysis in the joint t-f plane.

II. Gabor Transform for Characterization of Ultrasonic Gaussian Echoes

The Gabor transform (GT) of one ultrasonic echo signal, $s_1(t)$, with bandwidth parameter σ^2, time of arrival t_1, and frequency f_1

$$s_1(t) = \left(\frac{2\sigma^2}{\pi} \right)^{1/4} e^{-\sigma^2 (t - t_1)^2} e^{j 2\pi f_1 (t - t_1)} \tag{2}$$

can be derived as [3]

$$S_{1GT}(\tau, f) = C_1 e^{j 2\pi [f\tau/2 - f_1 t_1]} \cdot \int_{-\infty}^{\infty} e^{-\sigma^2 (t - t_1)^2 - \alpha^2 (t - \tau)^2} e^{-j 2\pi (f - f_1) t} dt \tag{3}$$

where C_1 is a constant given by

$$C_1 = \left(\frac{4\sigma^2 \alpha^2}{\pi^2} \right)^{1/4} \tag{4}$$

Thus the magnitude squared of the GT can be derived as

$$|S_{1GT}(\tau, f)|^2 = \frac{\sqrt{4\sigma^2 \alpha^2}}{\sigma^2 + \alpha^2} \exp\left[\frac{-2\sigma^2 \alpha^2 (\tau - t_1)^2}{\sigma^2 + \alpha^2} \right] \cdot \exp\left[\frac{-2\pi^2 (f - f_1)^2}{\sigma^2 + \alpha^2} \right] \tag{5}$$

If we have more than one echo in an ultrasonic signal then the GT of that signal generates energy not only at the time and frequency of occurrence of individual echoes (commonly known as auto-terms) but energy at different time and frequency may also be generated (commonly known as cross-terms).

The GT cross-terms of the two ultrasonic echoes occur at the intersection of two overlapping GT of individual echoes. If the cross-section, $\left|S_{1_{GT}}(\tau, f)\right| \cdot \left|S_{2_{GT}}(\tau, f)\right| = 0$, then there will be no cross-terms in GT of two ultrasonic echoes.

The GT cross-terms of two ultrasonic echoes can have a maximum magnitude of twice the product of the magnitude of the two GT. However these cross-terms occur at the intersection of the two GTs [4]. Therefore these cross-terms are far less in energy and are less important as far as the detection of times of arrival and frequencies of individual echoes are concerned.

The GT cross-terms of two ultrasonic echoes are modulated by a cosine function whose argument is a function of the difference between the phase of the GT of the two echoes which is dependent on the difference in center frequencies and center times the two echoes. This may be advantageous to perform post processing (filtering) to eliminate this modulation effect.

The above results can be generalized for multiple ultrasonic echoes and the GT of a signal containing N ultrasonic echoes

$$s(t) = \sum_{i=1}^{N} A_i \left(\frac{2\sigma_i^2}{\pi} \right)^{1/4} e^{-\sigma_i^2 (t - t_i)^2} e^{j2\pi f_i (t - t_i)} \quad (6)$$

where A_i, σ_i, t_i, and f_i are amplitude, sharpness, time of arrival and center frequency parameters for each echo, can be expressed as [3]

$$\left|S_{GT}(\tau,f)\right|^2 = \sum_{i=1}^{N} A_i^2 \left|S_{i\,GT}(\tau,f)\right|^2 + 2\sum_{j=1}^{N-1} \sum_{k=j+1}^{N} A_i A_k \left|S_{j\,GT}(\tau,f)\right| \left|S_{k\,GT}(\tau,f)\right| \cos\left[2\pi(f - f_i)\frac{\sigma_i^2 t_i^2}{\sigma_i^2 + \alpha^2} - 2\pi(f - f_k)\frac{\sigma_k^2 t_k^2}{\sigma_k^2 + \alpha^2}\right] \quad (7)$$

III. Time-Frequency Resolution of Gabor Transform for Ultrasonic Gaussian Echoes

The evaluation of time of arrival and center frequency of multiple echoes in an ultrasonic signal is dependent on the t-f resolution of the GT of the signal. These two paramters are very important in ultrasonic detection These are the time and frequency centroids of GT of the signal and can be evaluated for one ultrasonic echo of Equation (2) [3]:

In this paper we perform a quantitative evaluation of the affect on t-f resolution (spread) of GT. The t-f resolution of GT is affected by the effective width of the Gaussian window used in the evaluation of GT of a signal. The effective width of the GT window function is dependent on the parameter α^2. This parameter, α^2, plays an important role in displaying the energy of the signal in the joint t-f plane. GT has the optimum time-frequency concentration. But due to inherent t-f resolution trade-off it has broader time-frequency spread than WVD. The measure of energy concentration for the joint t-f representation is considered to be the quadratic t-f moment around the centroids (τ_s, f_s) [4]. If τ_s and f_s, time and frequency centroids, are defined as :

$$\tau_s = \int t . |s(t)|^2 dt, \qquad f_s = \int f . |S(f)|^2 df \quad (8)$$

Then the t-f concentration measure for GT of a single echo, expressed in Equation (9), can be calculated as:

$$\sigma^2_{|S_{GT}(\tau,f)|^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[(\tau - \tau_s)^2 + (f - f_s)^2\right] . \left|S_{GT}(\tau,f)\right|^2 d\tau\, df = \left[\frac{(\sigma^2 + \alpha^2)}{4\sigma^2\alpha^2} + \frac{(\sigma^2 + \alpha^2)}{4\pi^2}\right] \quad (9)$$

We have evaluated numerically the concentration measure for time and frequency for the GT of a single echo, which corresponds to the exact solution. The exact solutions of time & frequency concentration of a Gaussian echo for different values of effective width parameter (σ^2) verses GT window effective width parameter, α^2, are plotted in Figures (1a-1e). The ideal t-f concentration of a Gaussian echo is depicted by Wigner-Ville distribution (WVD), which is exactly the same as that of the actual echo. The optimum resolution for the GT of a single echo can be obtained if the window parameter $\alpha^2 = \sigma^2$ [5]. This is also confirmed by substituting $\alpha^2 = \sigma^2$ in Equation (9). The optimum concentration of GT of a single echo is given by the following equation.

$$\sigma^2_{|S_{GT}(\tau,f)|^2_{Optimum}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[(\tau - \tau_s)^2 + (f - f_s)^2\right] . \left|S_{GT}(\tau,f)\right|^2 d\tau\, df = \left[\frac{1}{2\sigma^2} + \frac{\sigma^2}{2\pi^2}\right] = 2\left[\frac{1}{4\sigma^2} + \frac{\sigma^2}{4\pi^2}\right] \quad (10)$$

The optimum values of time and frequency concentrations of GT of a Gaussian echo are twice those of the WVD of a Gaussian echo. Figures (1a-1e) also show the ideal and optimum time and frequency concentrations. These figures indicate that the minimum (time + frequency) resolution for GT of a Gaussian echo is obtained at α^2 (GT window effective width parameter) = **3.14,** irrespective of the Gaussian echo effective width parameter, σ^2. The minima of (Time + Frequency) resolution curve of GT of a Gaussian echo occurs at α^2 (GT window effective width parameter) = **3.14,** numerically as well as analytically. If α^2 (GT window effective width parameter) = **3.14** = π, then the minimum concentration measure of GT of a Gaussian echo is given by:

$$\sigma^2_{|S_{GT}(\tau,f)|^2_{minimum}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[(\tau-\tau_s)^2+(f-f_s)^2\right]$$

$$\cdot|S_{GT}(\tau,f)|^2\,d\tau\,df\Big|_{\alpha^2=\pi} = \left[\frac{1}{4\sigma^2}+\frac{1}{4\pi}+\frac{\sigma^2}{4\pi^2}+\frac{1}{4\pi}\right]$$

$$= \left[\frac{1}{4\sigma^2}+\frac{\sigma^2}{4\pi^2}+\frac{1}{2\pi}\right] \quad (11)$$

The minimum values of time and frequency concentrations of GT of a Gaussian echo are each 0.079577 ($1/4\pi$) more than those of the WVD of a Gaussian echo (or actual time and frequency concentration of the Gaussian echo). Gaussian echo bandwidth approximately varies from 1 MHz to 3 MHz. Therefore, GT of the signal with minimum (time + frequency) concentration of Gaussian echoes can be obtained by taking GT window effective width parameter, α^2, equal to π without the knowledge of the echo effective width parameter, σ^2. In the case of signal containing multiple echoes the minimum or the optimum GT concentration may not be the reasonable choice for detection purposes. The apriori knowledge of the transducer's characteristics, along with the GT (time + frequency) resolution curves of Figure (1), and the following table are useful in choosing the proper value of GT window effective width parameter. This choice is for better detection of time of arrival and center frequency of each echo in multiple echoes. Figures (1a-1e) indicate that the value of α^2 at the knee of the (time + frequency) resolution curve is a better trade-off for detection of multiple echoes both in time and frequency domain. The following Table (1) states the time, frequency and time + frequency resolutions for ideal (actual Gaussian and its WVD), optimum (GT for $\alpha^2 = \sigma^2$), and for better detection (α^2 at the knee of time + frequency resolution curve) cases.

IV. Results and Discussions

We have numerically evaluated the GT of a simulated signal, which contains two ultrasonic echoes. These echoes are located at a close proximity of each other in time and frequency. Figures (2a-2e) show this signal and surface plots of GT of this signal with different GT window parameters. The results of Table (1) are confirmed by Figures (2a-2e) for echo bandwidth of approximately 1 MHz (σ^2 = **0.4**). These figures indicate that WVD of the signal containing two echoes of same bandwidth with different frequencies, and located in close proximity of each other, has the ideal concentration for both echoes but the cross-terms obscure the actual echoes. The optimum GT of the signal ($\alpha^2 = \sigma^2$) has better frequency concentration but time spread is more, which will make the detection of closely, located echoes difficult. The minimum resolution GT ($\alpha^2 = \pi$) of the same signal has better time concentration but the frequency spread is more which will make the detection of echoes with close frequencies difficult. The trade-off is that by the apriori knowledge of the system's characteristics we can use the time + frequency resolution curves to get the value of GT window parameter, α^2, for better detection of multiple echoes in both time and frequency.

We have applied GT to the experimental ultrasonic data. This experimental signal is shown in Figure (3a). It contains a defect echo embedded in grain echoes. The frequency of defect echo is very near to that of the grain echoes. Figures (3b) depict the mesh plot of minimum resolution GT ($\alpha^2 = \pi$) of the experimental data, respectively. These plots indicate the better detection of defect echo in time but the frequency concentration of the defect as well as grain echoes is not good. Figures (3c) depict the mesh plot of GT of experimental data when GT widow parameter α^2 = **2**. This choice gives a little better resolution in frequency at the expanse of time resolution. Much better resolution in frequency can be obtained by taking α^2 = **1** as depicted by mesh plot of GT of the signal in Figures (3d). Thus the apriori knowledge of the system (transducer) characteristics and the use of time + frequency resolution curves will help choose the proper value of α^2 for better time-frequency concentration and detection of time of arrival and center frequency of multiple echoes.

IV. Conclusions

In this paper we performed a quantitative evaluation of the affect on t-f resolution (spread) of GT. The t-f resolution of GT is affected by the effective width of the Gaussian window used in the evaluation of GT of a signal. The effective width of the GT window function is dependent on the parameter α^2. This parameter, α^2, plays an important role in displaying the energy of the signal in the joint t-f plane. The optimum values of time and frequency concentrations of GT of a Gaussian echo are twice those of the WVD of a Gaussian echo due to inherent t-f resolution trade-off. We have generated a set of curves from the analytical solution that depict t-f resolution (concentration) of a Gaussian ultrasonic echo for different values of GT window duration. With the apriori knowledge of measurement system's characteristics and using these t-f concentration curves we can choose an optimal GT window parameter for better detection of multiple echoes in both time and frequency.

VI. REFERENCES

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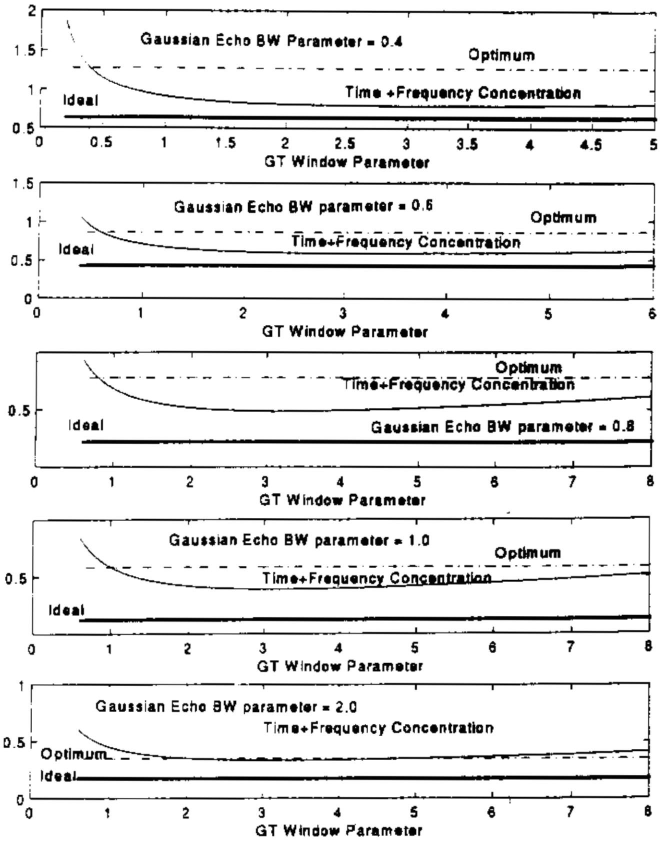
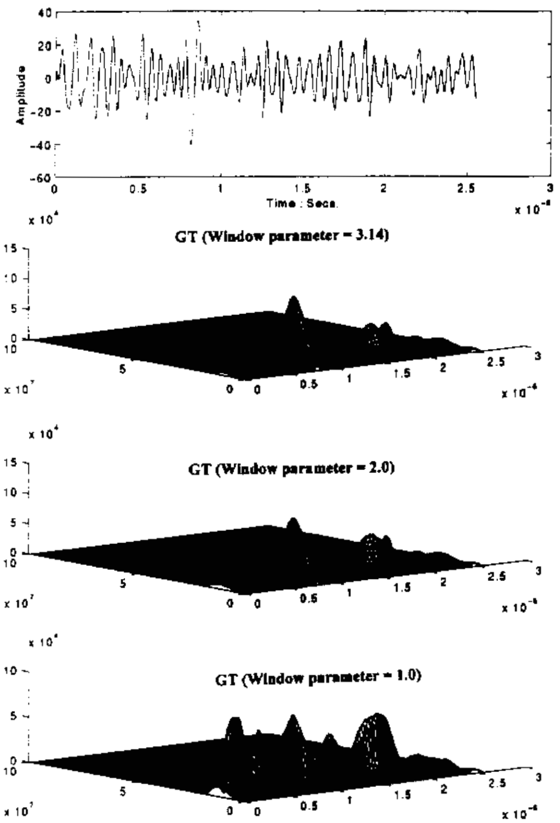


Figure 1. Time-Frequency Concentration Curves for Different GT Window Parameters



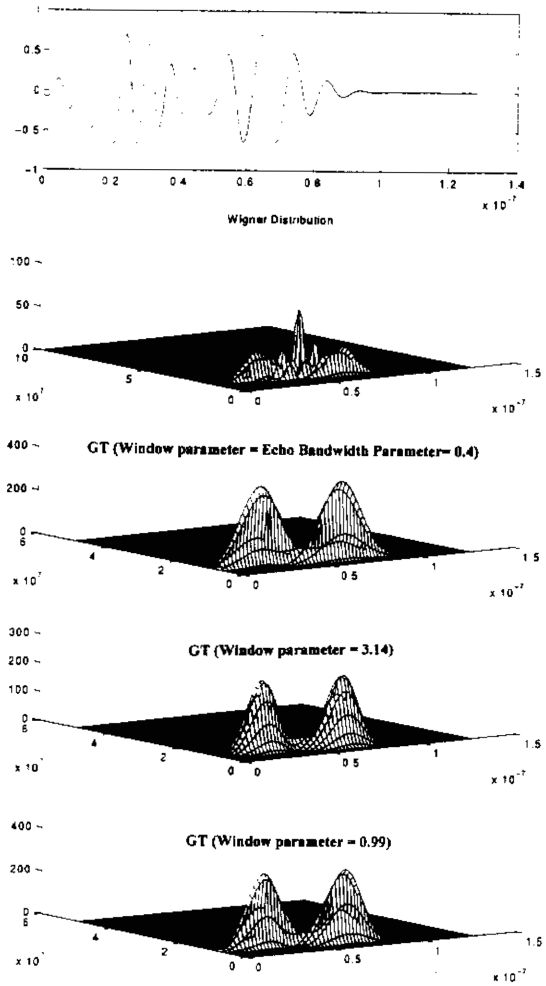


Figure 2. Simulated Two Ultrasonic Echoes, WVD, and GT with Different Window Parameters

Table-1. Time and Frequency Concentration for Gaussian Echo

Gaussian Echo Bandwidth parameter σ^2	Actual (WVD) Resolution $\left[\frac{1}{4\sigma^2}+\frac{\sigma^2}{4\pi^2}\right]$			GT Optimum Resolution GT Window Parameter, $\alpha^2=\sigma^2$ $\left[\frac{1}{2\sigma^2}+\frac{\sigma^2}{2\pi^2}\right]$			GT Minimum Resolution $(\alpha^2=\pi)$ Eff. Width = 35 $\left[\frac{1}{4\sigma^2}+\frac{\sigma^2}{4\pi^2}+\frac{1}{2\pi}\right]$			GT Resolution for Better Detection (α^2 at Knee of Resolution Curve)		
(Effective Time Width)	Time	Freq	Sum	Time	Freq	Sum	Time	Freq	Sum	Time	Freq	Sum
0.4 (95)	0.62	0.01	0.63	1.24	0.02	1.26	0.70	0.09	0.79	0.87	0.03	0.90
0.6 (79)	0.42	0.01	0.43	0.84	0.02	0.86	0.496	0.095	0.59	0.62	0.05	0.67
0.8 (67)	0.31	0.02	0.33	0.62	0.04	0.66	0.39	0.10	0.49	0.47	0.06	0.53
1.0 (61)	0.25	0.03	0.27	0.50	0.06	0.56	0.33	0.10	0.43	0.37	0.08	0.45
2.0 (43)	0.12	0.05	0.17	0.24	0.10	0.34	0.20	0.13	0.33	0.20	0.13	0.33

Figure 3. Experimental Ultrasonic Signal of a Flaw and grain echoes, and GT of signal with Different Window Parameters