Improved Time-Frequency Distribution using Singular Value Decomposition of Choi-Williams Distribution

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***Abstract-* The Choi-Williams distribution (CWD) is an effective time-frequency (TF) distribution to suppress cross-terms. However, one notable drawback of the CWD is that it fails to suppress cross-terms when either center frequencies of two components are close or their time overlap is significant. In this paper, this problem is addressed by reconstructing TF distribution using basis functions which are extracted through Singular Value Decomposition (SVD) from CWD. The numerical analysis of multi-component Gaussian echoes are presented by using SVD of CWD. It is found that this decomposition and reconstruction approach efficiently eliminate residual cross-terms for which the CWD failed to remove. Results are presented in support of exhibiting the effectiveness of SVD of CWD for cross-term suppression.**

I. INTRODUCTION

There are several well-known time-frequency distributions, the Short-Time Fourier Transform (STFT) [1,2], the Gabor transform, the wavelet transform[3], the Wigner distribution [4,5], and some modified Wigner distributions [6-9]. None of these time-frequency distributions are perfect. The STFT has the advantage of low computational complexity, but it has limited time-frequency (TF) resolution. The wavelet transform has the ability to display high resolution in time at higher frequencies. The Wigner distribution can furnish the highest TF clarity but certain spurious information which is called cross-terms can damage the clarity. It is desirable to design a TF distribution with high resolution, lower computational complexity and without spurious cross-terms. This is why modified Wigner distributions, like the Gabor Wigner transform is introduced, and why the Choi-Williams Distribution (CWD) [10], a Wigner distribution with an exponential kernel, is popular and has been used widely in signal processing [11].

Cross-terms are the spurious information caused by the bilinear property of Wigner distribution, while auto-terms are the desired TF information of signals. The CWD is a very effective distribution in diminishing cross-terms while retaining auto-terms. However, the CWD becomes worse and diagnosing the useful information from it becomes difficult when the two components' center frequencies are close or even heavily overlapped. Therefore, it is necessary to find an algorithm to improve the quality of CWD.

This paper proposes a new algorithm of TF distribution to address the limited performance of CWD for cross-term suppression. This algorithm is using Singular Value Decomposition (SVD) to extract basis functions of multi-component signals to reconstruct time-frequency distribution [12]. We call it SVD of CWD (CWD-SVD). The CWD-SVD algorithm is able to remove the cross-terms successfully when the center frequencies of two components are near or the overlap is significant. Furthermore, it is able to separate two components with near center frequencies that are totally overlapped when significant noise is represented.

II. SINGULAR VALUE DECOMPOSITION OF CHOI-WILLIAMS DISTRIBUTION

We study the SVD of CWD in multi-component signals that are composed of Gaussian echoes (or Gaussian wavelets). In radar, sonar, seismic, ultrasonic nondestructive detection or medical diagnosis application areas, the Gaussian echo is a primary model in signal decomposition, compression and denoising. The frequency and amplitude of signals in these areas vary with time. Hence, TF distributions are compelling tools to deal with these time-varying non-stationary signals. Therefore, there is a practical demand to start with the study of multi-component Gaussian echoes in the analysis of singular value decomposition of Choi-Williams distribution.

We assume the Gaussian echo model as [13]

$$s_1(t) = \left(\frac{2\alpha}{\pi}\right)^{1/4} e^{-\alpha(t-t_0)^2 + j\omega_0 t} \quad (1)$$

where t_0 is the arrival time, ω_0 is the center frequency and α is the bandwidth factor. Thus,

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according to the Choi-Williams formula, the CWD of Gaussian echo is [11]:

$$\mathrm{CWD}(t,\omega) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\sigma+8\alpha\tau^2}} e^{\left(-\frac{2\alpha\sigma(t-t_0)^2}{\sigma+8\alpha\tau^2}\right)} e^{\left(-j\tau(\omega-\omega_0)-\frac{\alpha\tau^2}{2}\right)} d\tau \qquad (2)$$

Considering a multi-component signal consisting of multiple Gaussian echoes, and based on the SVD theorem [12], we obtain the SVD decomposition equation as

$$\lambda_{iN} U_{iN}(\omega) = \int_{-\infty}^{+\infty} CWD_{s_N}(t-\tau,\omega) V_{iN}(\tau) d\tau \qquad (3)$$

where λ_{iN}, U_{iN} and V_{iN} are the i^{th} singular value, i^{th} basis functions of CWD_{s_N} which is CWD of the N-component signal for a given σ value ranging between 0.1 and 10.

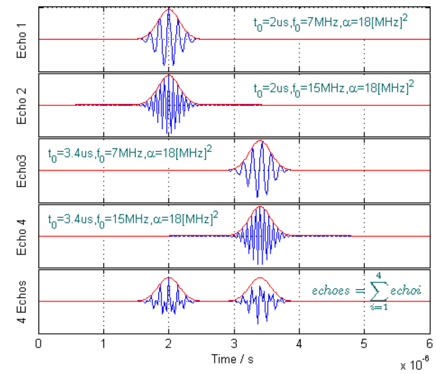
The CWD without cross-terms can be represented by λ_{jN}, U_{jN} and V_{jN} as

$$CWD(t,\omega) = \sum_{j=1}^{M} \lambda_{jN} U_{jN}(\omega) V_{jN}(t) \qquad (4)$$

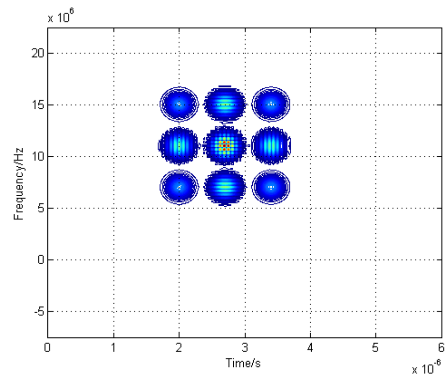
where λ_{jN}, U_{jN} and V_{jN} are the j^{th} singular value, j^{th} basis functions of N-component Gaussian echoes, and M is the total number of basis functions representing the auto-terms. Singular values and basis functions of cross-terms are discarded for improved time-frequency representation.

The characteristics of the TF representation using Equation 4 are demonstrated in the Figure 1. Figure 1a shows a four-component Gaussian echoes with both time and frequency overlaps. Figure 1b displays the undesirable cross-terms of the TF distribution using the Wigner distribution. As shown in Figure 1c, the CWD with σ =1 eliminates certain cross-terms; however, it failed to suppress the cross-terms between the two neighboring echoes which either have the same arrival times or have the same center frequencies. But, as shown in Figure 1d, these remaining cross-terms in CWD are eliminated totally by CWD-SVD.

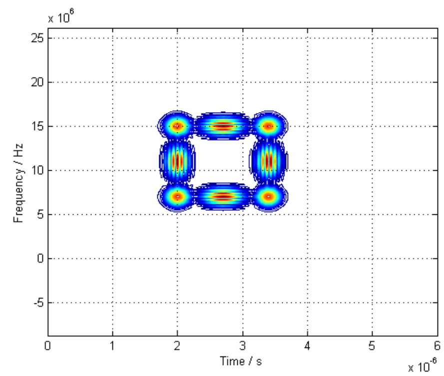
The following is a number of experiments to further verify the efficiency of CWD-SVD method. We simulated two echoes. Consider these echoes are moving toward each other with parameters 2[MHz]^2 (bandwidth), 4MHz and 5MHz center frequencies. Signals are simulated using 100 MHz sampling rate. Figure 2 shows two Gaussian echoes when the overlap is 0%. The red arrows denote the moving direction of two echoes. The terms t2 and t1 as shown in Figure 2 denote the arrival time of these echoes.



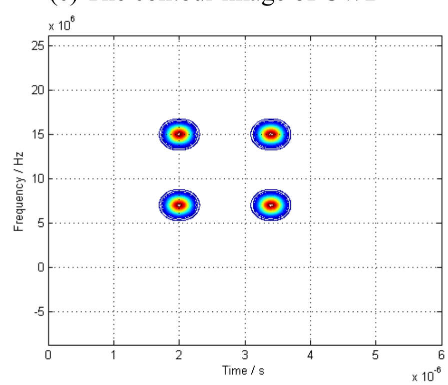
(a) Four-component Gaussian echoes



(b) The contour image of Wigner Distribution



(c) The contour image of CWD



(d) The contour image of CWD-SVD

Figure 1. Contour image of CWD and CWD-SVD of four Gaussian echo components

We define the percentile echo overlap index μ as [14],

$$\mu = \left(1 - \frac{|t_1 - t_2|}{3(\sigma_1 + \sigma_2)}\right)\% \qquad (5)$$

where t_1 and t_2 are the arrival time of two Gaussian echoes, σ_1 and σ_2 are the two echoes' spreading parameters. We assume that $|t_1 - t_2| \leq 3(\sigma_1 + \sigma_2)$ in this formula and when $|t_1 - t_2| > 3(\sigma_1 + \sigma_2)$ we assume that there is no overlap. Figure 3 shows two echoes with noise (SNR=0 dB) and 95% overlap. Compared with CWD, CWD-SVD improves the quality of estimation. This can clearly be observed by inspecting the estimation values shown in Table 2. Furthermore, Figure 4 and Table 2 present estimation of arrival times of the second echo relative to the first echo when overlap between these echoes ranging from 0%, 35%, 65%, 85%, 90%, 95%, 97%, to 100% with different SNR values: 20 dB, 10dB, 3dB and 0dB respectively. From Table 2, we calculate that the relative error between the estimation value and the reference value is no more than 2% (See Figure 4a). This is a very small estimation error and it shows that the algorithm of CWD-SVD is very useful in the requirement of high resolution estimation applications in the presence of noise.

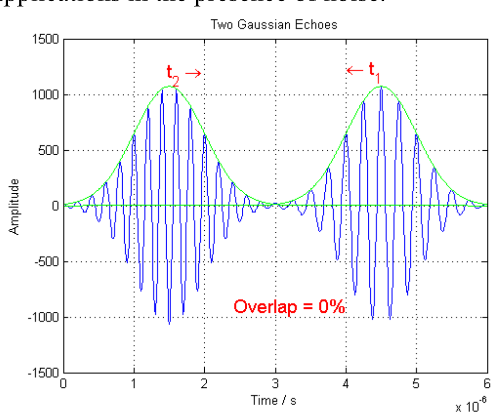


Figure 2. Two Gaussian echoes when no overlap

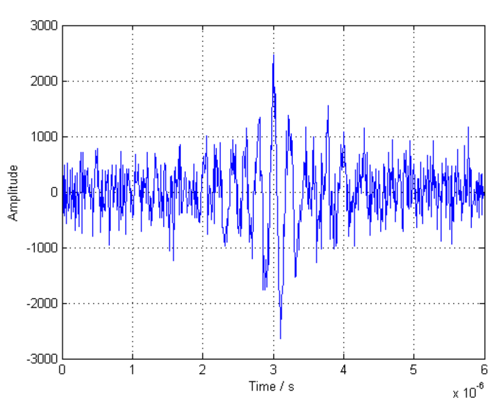
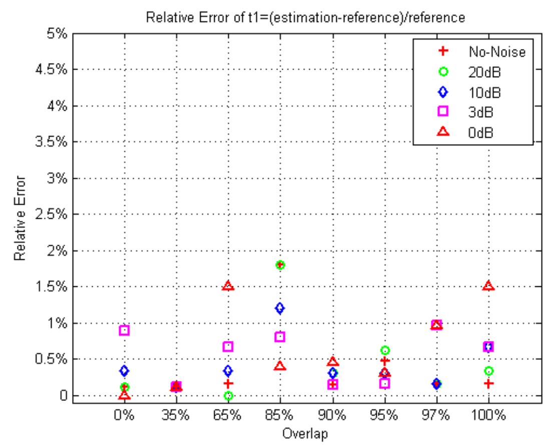


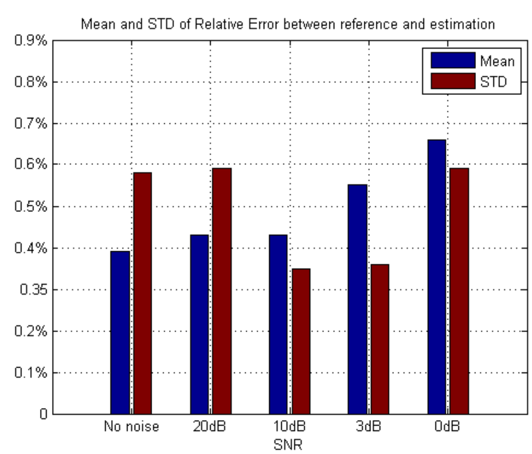
Figure 3. The estimation results of two Gaussian echoes with 95% overlap when SNR=0dB

Table 1. Estimated arrival times and center frequencies of two Gaussian Echoes when SNR= 0dB and overlap = 95%

	Reference	CWD estimate	CWD-SVD estimate
Arrival Time Echo 1 (μs)	3.2	3.1800	3.1900
Center Freq Echo 1 (MHz)	4	4.0039	4.0039
Arrival Time Echo 2 (μs)	3	2.9400	2.9600
Center Freq Echo 2 (MHz)	5	4.9805	4.9316



(a) The relative error of the arrival time of the late arrival echo



(b) Mean(left bar) and STD (right bar) for different SNRs

Figure 4. Mean and STD of relative error between the reference and estimation values

Table 2. Estimation of arrival times for different SNRs and different degrees of overlap

Overlap	Ref. time (μs)	The estimated arrival time (μs)				
		No noise	20dB	10dB	3dB	0dB
0%	4.50	4.505	4.505	4.515	4.460	4.500
35%	4.50	4.505	4.505	4.495	4.505	4.505
65%	3.00	2.995	3.000	3.010	3.020	3.045
85%	2.50	2.455	2.455	2.470	2.480	2.490
90%	3.30	3.295	3.290	3.310	3.305	3.285
95%	3.20	3.185	3.180	3.190	3.195	3.190
97%	3.10	3.095	3.095	3.105	3.070	3.070
100%	3.00	3.005	3.010	3.020	2.980	3.045
Mean	***N/A***	***0.39%***	***0.43%***	***0.43%***	***0.55%***	***0.66%***
STD	***N/A***	***0.58%***	***0.59%***	***0.35%***	***0.36%***	***0.59%***

III. CONCLUSION

In this paper, a new time-frequency distribution is proposed. We call it Singular Value Decomposition of Choi-Williams Distribution (CWD-SVD). The SVD allows CWD to be expressed in terms of the summations of the basis functions and the singular values to achieve the elimination of cross-terms. Using this characteristic, we can reconstruct the time-frequency representations without cross-terms by selecting only basis functions and singular values, which correspond to the true information of multi-component signals. Based on numerical analysis, we see this algorithm is robust and offers the ability to estimate the parameter of signal components precisely, even in the presence of the noise. Therefore, this is a promising and feasible method to remove spurious information in CWD, and it is able to keep the high resolution of the Wigner distribution simultaneously.

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